

CME 100 Fall 2009 Lecture Notes

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Reading: Thomas - 12.1, 12.2, 12.3 and 12.4

1 Vectors in 2D and 3D

1.1 Definition of vectors

Many times in engineering, one wants to model quantities that are not adequately described by a single number, like temperature or pressure, but rather by a *direction* and *magnitude*. These are called **vector quantities** or simply **vectors**. Examples of vector quantities are:

- Force: it has a magnitude, the strength, and a direction.
- Velocity: it also has a magnitude, the speed, and a direction.
- Acceleration.
- Electric and magnetic fields.
- A trajectory when moving along a straight line for some time t .

1.2 Notation

If a quantity is a vector, then it is either going to be in boldface, such as \mathbf{u} or have an arrow over it, such as \vec{u} .

1.3 Geometric interpretation of a vector

A vector \vec{u} has a direction and a magnitude. A convenient geometrical representation of vector is a straight line segment drawn in space in the direction of the vector, with an arrowhead indicating its sense. The length of the line segment is given by the magnitude of the vector. This is sometimes called a **directed straight line segment**.

In this course, we will primarily use the Cartesian coordinate system to represent vectors analytically, that is using numbers. Let's assume that we draw a vector \vec{u} starting from the origin. It will end at some point P with Cartesian coordinates (u_1, u_2, u_3) . By definition, these are the **Cartesian coordinates** of vector \vec{u} . For example, on Figure 1, the vectors $\vec{u} = (4, 3)$ and $\vec{v} = (2, 3, 1)$ are shown. The magnitude or length of \vec{u} is given by:

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

1.4 Vector operations

We can define operations on vectors as with scalars, but one has to be a little more careful.

Scalar multiplication for a vector $\vec{u} = (u_1, u_2, u_3)$ is defined as

$$k\vec{u} = (ku_1, ku_2, ku_3)$$

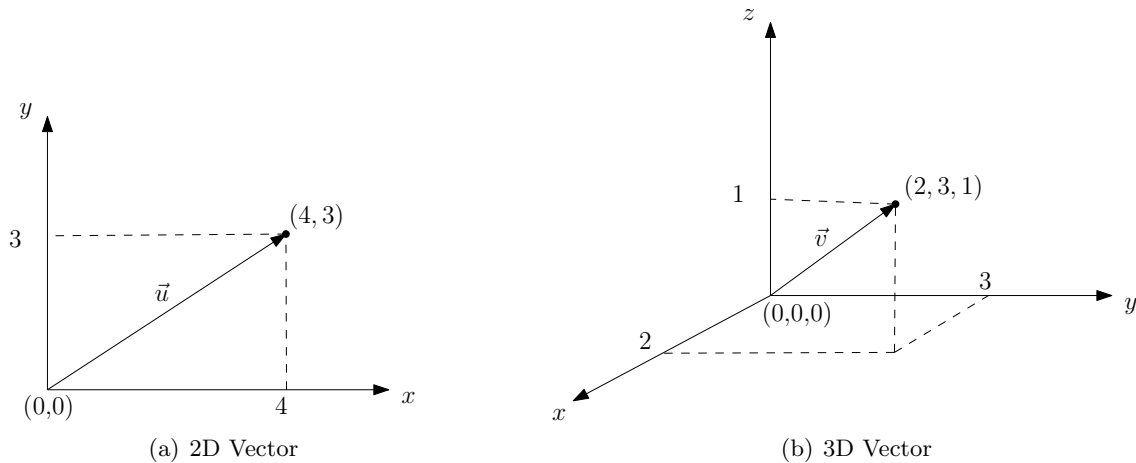


Figure 1: Geometric Interpretation of a vector

Multiplying a vector with a scalar does not change its direction, but it changes the length of the vector. To see this, note that $k\vec{u} = (ku_1, ku_2, ku_3)$ and the length is now given as

$$\begin{aligned} |k\vec{u}| &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\ &= |k|\sqrt{u_1^2 + u_2^2 + u_3^2} \\ &= |k| |\vec{u}| \end{aligned}$$

A common mistake in computing the length of a vector is to forget the absolute value, $|\cdot|$, around k . Remember that length cannot be negative!

The effect of multiplying a vector \vec{u} with scalars 2.5, -1 and 0.3 is given in Figure 2 in different colors. Note that the different vectors all lie on top of each other as scalar multiplication of a vector cannot change the direction of the vector, except for reversing it. But scalar multiplication does change the magnitude of \vec{u} !

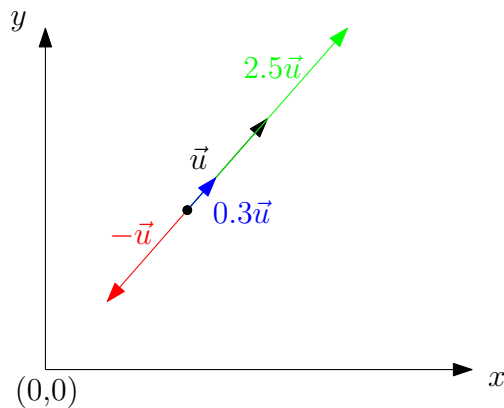


Figure 2: Scalar multiplication of a vector

Vector addition for two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is defined as

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

As with other operations, vector addition also has a nice geometric interpretation and can be seen via the well known *Parallelogram Law* or the *Triangle Law*. These are depicted in Figure 3.

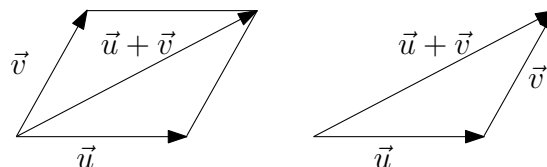


Figure 3: The parallelogram law and the triangle law of vector addition. The parallelogram law states that if one draws a parallelogram with \vec{u} on one side and \vec{v} on the other then $\vec{u} + \vec{v}$ is given by the diagonal of the parallelogram. The triangle law states that if you draw vector \vec{u} and vector \vec{v} starting at the end of \vec{u} , and then join the first and last points, you obtain vector $\vec{u} + \vec{v}$.

1.5 Standard basis

A 3D vector can be conveniently represented using the *standard basis*:

$$\vec{i} = (1, 0, 0)$$

$$\vec{j} = (0, 1, 0)$$

$$\vec{k} = (0, 0, 1)$$

Each of these vectors \vec{i} , \vec{j} and \vec{k} has only one non-zero component out of three. Any vector $\vec{u} = (u_1, u_2, u_3)$ can be written using the standard basis as

$$\begin{aligned} \vec{u} &= (u_1, u_2, u_3) \\ &= (u_1, 0, 0) + (0, u_2, 0) + (0, 0, u_3) \\ &= u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) \\ &= u_1\vec{i} + u_2\vec{j} + u_3\vec{k} \end{aligned}$$

Note each of these standard basis vectors points in the direction of one axis in the Cartesian coordinate system.

1.6 Displacement Vector

The displacement vector \overrightarrow{PQ} defines a vector starting at point $P = (x_1, y_1, z_1)$ ending at a point $Q = (x_2, y_2, z_2)$ and is shown in Figure 4. As with any other vector, \overrightarrow{PQ} has a direction and a magnitude (or length), and in this case the direction is the direction going from P to Q , and the magnitude is the distance between P and Q . The coordinates of the displacement vector are:

$$\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

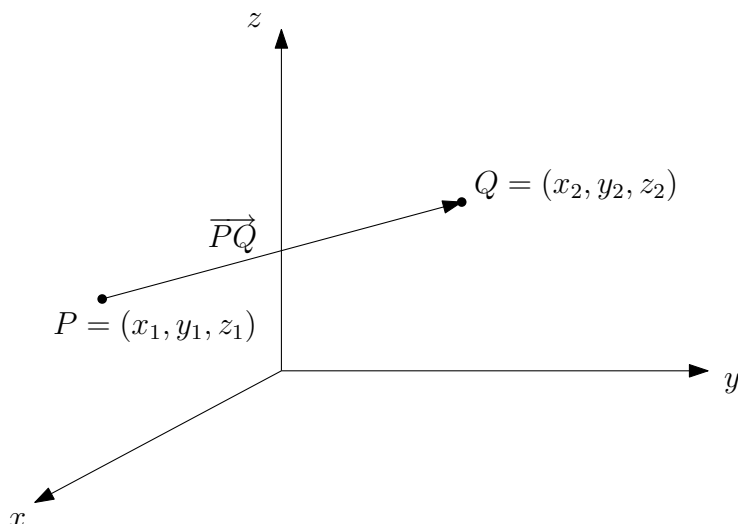
The distance between P and Q is equal to the length of \overrightarrow{PQ} and therefore is given by:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 1. Given two points $P = (1, -1, 2)$ and $Q = (-1, 1, 1)$, compute the distance between the two points.

First, let us compute the displacement vector \overrightarrow{PQ} :

$$\overrightarrow{PQ} = (-1 - 1, 1 - (-1), 1 - 2) = (-2, 2, -1)$$

Figure 4: The displacement vector from P to Q

Then, the length of \vec{PQ} is given by

$$|\vec{PQ}| = \sqrt{(-2)^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$$

Thus, the distance between P and Q is 3.

1.7 Unit Vector

We define \vec{u} to be a unit vector if \vec{u} has unit length, i.e., $|\vec{u}| = 1$.

Every vector \vec{u} can be decomposed as a length times a unit vector and this can be seen as follows:

$$\vec{u} = |\vec{u}| \left(\frac{\vec{u}}{|\vec{u}|} \right)$$

The vector $\vec{u}/|\vec{u}|$ has unit length since:

$$\left| \frac{\vec{u}}{|\vec{u}|} \right| = \frac{1}{|\vec{u}|} |\vec{u}| = 1$$

The unit vector $\frac{\vec{u}}{|\vec{u}|}$ denotes the direction of \vec{u} and the magnitude of \vec{u} is given by $|\vec{u}|$.

Example 2. Given the vector $\vec{u} = (3, -1, 2)$, find its length and direction.

We can compute the length of \vec{u} as

$$|\vec{u}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$$

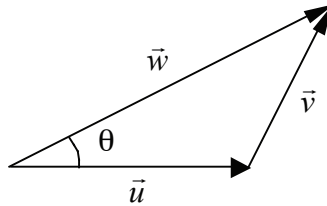
Thus, the unit vector along \vec{u} or direction of \vec{u} , is given by:

$$\frac{\vec{u}}{|\vec{u}|} = \left(\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right)$$

Thus, we can write the vector itself as its length times the direction:

$$\vec{u} = \sqrt{14} \cdot \left(\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right)$$

Example 3. An airplane flying due east at 500 mph in still air encounters a 70 mph tailwind acting in the direction 60° north of east. Determine the airplanes ground speed and direction.



Matlab code:

```
% Define vectors
% u velocity of plane with respect
%   to wind
% v velocity of wind with respect to
%   ground
u=[500,0]
u =
    500     0
v=[70*cos(60*pi/180), 70*sin(60*pi/180)]
v =
    35.0000    60.6218
% Add u and v
w=u+v
w =
    535.0000    60.6218
% Find magnitude of w
speed=norm(w)
speed =
    538.4236
% Determine direction
theta=atan(w(2)/w(1))*180/pi
theta =
     6.4647
```

2 Dot Product

The dot product of two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is defined as

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Note that the dot product of two vectors always results in a *scalar*.

2.1 Properties

A few properties of the dot product are:

1. The dot product of a vector with itself gives the square of its magnitude:

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2$$

2. The dot product is commutative:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

3. The dot product is distributive:

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

2.2 Geometric interpretation of the dot product

Recall that the law of cosines for two vectors states that (see Figure 5):

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta \quad (1)$$

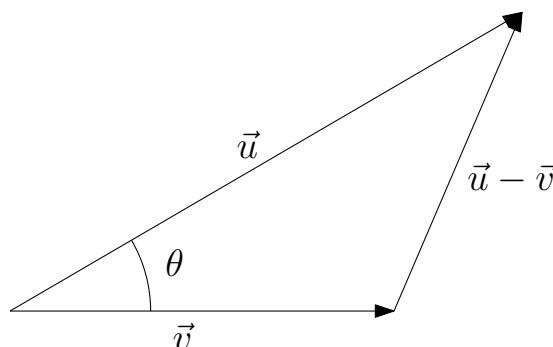


Figure 5: The law of cosines

Using the properties of the dot product, we can calculate $|\vec{u} - \vec{v}|^2$ in the following manner:

$$\begin{aligned} |\vec{u} - \vec{v}|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= |\vec{u}|^2 + |\vec{v}|^2 - 2(\vec{u} \cdot \vec{v}) \end{aligned} \quad (2)$$

Comparing Equation (1) and (2), we have

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta}$$

Thus, the angle between 2 vectors \vec{u} and \vec{v} can be computed as:

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$$

Note that this directly implies that:

If $\vec{u} \cdot \vec{v} = 0$, then \vec{u} and \vec{v} are perpendicular.

Example 4. Given $\vec{u} = (1, 3)$ and $\vec{v} = (-1, 1)$, find the angle between them.

We have $|\vec{u}| = \sqrt{1^2 + 3^2} = \sqrt{10}$. Similarly, $|\vec{v}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. Also, we can compute $\vec{u} \cdot \vec{v} = 1 \cdot (-1) + 3 \cdot (1) = 2$. Then the angle between the vectors is given by

$$\theta = \arccos\left(\frac{2}{\sqrt{10} \cdot \sqrt{2}}\right) = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx 63^\circ$$

2.3 Projections

By projection of \vec{u} along \vec{v} , we mean the component of \vec{u} in the direction of \vec{v} . The scalar component of \vec{u} in the direction of \vec{v} is given by:

$$|\vec{u}| \cos \theta = \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|}$$

We can also define the vector projection of \vec{u} onto \vec{v} as (see Figure 6):

$$\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \right) \cdot \left(\frac{\vec{v}}{|\vec{v}|} \right) = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$$

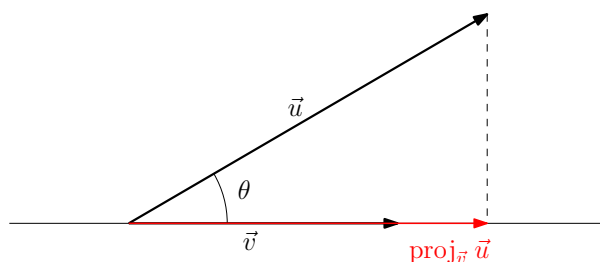


Figure 6: Vector projection of \vec{u} onto \vec{v} .

The component of \vec{u} perpendicular to \vec{v} can be computed by subtracting from \vec{u} , the component of \vec{u} along \vec{v} :

$$\vec{u} - \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$$

2.4 Physical application of the dot product

The work done by a force \vec{F} acting on a particle that moves in a straight line in the direction of \vec{d} and by a distance $|\vec{d}|$ is given by:

$$W = |\vec{F}| d \cos \theta$$

where θ is the angle between the force vector and the displacement vector \vec{d} (see figure 7). The work W can therefore be expressed in a simpler way using the dot product:

$$W = \vec{F} \cdot \vec{d}$$

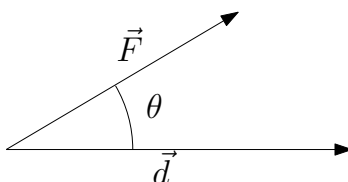


Figure 7: Work done by the force \vec{F} .

Example 5. A child pulls a sled up a hill with force $\vec{F} = (8, 2)$ (in pounds) and displacement $\vec{d} = (100, 20)$ (in feet), as shown in figure 8. Find the work done by the child.

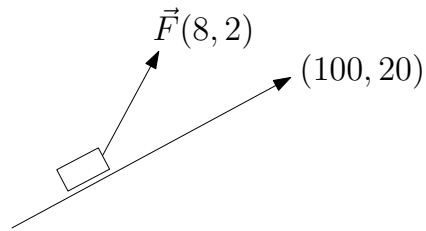


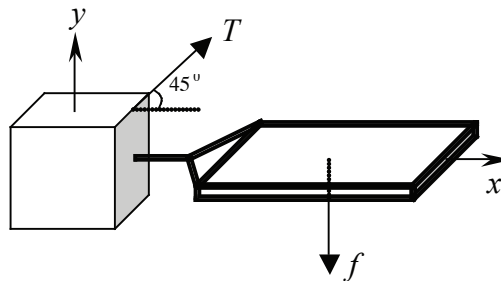
Figure 8: Work done by a child pulling a sled uphill

Using the expression derived above, we have the work done as

$$W = (8, 2) \cdot (100, 20) = 840 \text{ foot-pound}$$

Example 6. A force of 10 N is exerted by a thruster on a satellite at 45° with respect to the horizontal. Exhaust particles hit the solar array surface producing a drag force of 0.5 N pointing in the direction normal to the surface of the array.

1. Determine the loss in thrust due to drag
2. Determine the component of drag in the direction perpendicular to the direction of thrust



Matlab code:

```
% Define vectors
T=[10*cos(45*pi/180), 10*sin(45*pi/180)]
T =
    7.0711    7.0711
f=[0,-0.5]
f =
    0   -0.5000
% Find projection of f in the direction of T
proj_T_f=dot(f,T)/(norm(T)^2)*T
proj_T_f =
   -0.2500   -0.2500
% Find magnitude of projection
norm(proj_T_f)
ans =
    0.3536
% Find effective thrust level
Thrust=10-norm(proj_T_f)
```



```

Thrust =
    9.6464
% Find the component of f perpendicular to T
f_perp_T=f-proj_T_f
f_perp_T =
    0.2500   -0.2500

```

3 Cross product

Definition: Given \vec{u} and \vec{v} , we define their cross product as

$$\vec{u} \times \vec{v} = (u_2v_3 - v_2u_3, u_3v_1 - v_3u_1, u_1v_2 - v_1u_2)$$

A simple mnemonic to remember the cross product formula can be seen in Figure 9 which says that to compute the first component of $\vec{u} \times \vec{v}$, we use the second component of \vec{u} and the third component of \vec{v} and so on.

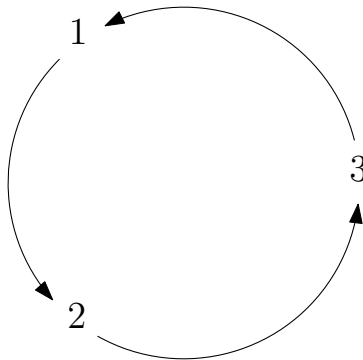


Figure 9: A mnemonic to remember the cross product formula

3.1 Geometric interpretation

The cross product $\vec{u} \times \vec{v}$ is a vector which is perpendicular to the plane containing vectors \vec{u} and \vec{v} , as shown in figure 10. This can be confirmed by checking that the following dot products compute to 0:

$$\begin{aligned} \vec{u} \cdot (\vec{u} \times \vec{v}) &= 0, \text{ therefore } \vec{u} \text{ is } \perp \text{ to } \vec{u} \times \vec{v} \\ \vec{v} \cdot (\vec{u} \times \vec{v}) &= 0, \text{ therefore } \vec{v} \text{ is } \perp \text{ to } \vec{u} \times \vec{v}. \end{aligned}$$

Note that there are two vectors that are perpendicular to the plane containing \vec{u} and \vec{v} , one pointing up and one down. Among these two, we choose the cross product direction according to the *Right Hand Rule* which states that if you curl the fingers of your right hand going from \vec{u} to \vec{v} then the thumb points in the direction of $\vec{u} \times \vec{v}$.

The length of the cross product, i.e., $|\vec{u} \times \vec{v}|$, is equal to:

$$|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$$

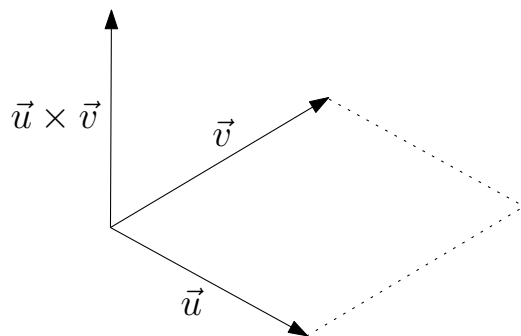


Figure 10: The cross product

This is the same as the area of the parallelogram formed by the two vectors. The area of the triangle formed by \vec{u} and \vec{v} is half the area of the parallelogram and hence given by $\frac{1}{2}|\vec{u} \times \vec{v}|$.

Example 7. We will show in this example the formula with $\sin \theta$:

1. Show that: $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$

2. Using the above identity, show that: $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$

Question 1 can be proved by direct calculation. Here is a sketch of the proof. The cross product is given by:

$$\vec{u} \times \vec{v} = (u_2v_3 - v_2u_3, u_3v_1 - v_3u_1, u_1v_2 - v_1u_2)$$

From this equation $|\vec{u} \times \vec{v}|^2$ can be computed. Using the following three equations:

$$|\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2$$

$$|\vec{v}|^2 = v_1^2 + v_2^2 + v_3^2$$

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

we can show that $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$.

Question 2 is proved using trigonometry:

$$\begin{aligned} |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 &= |\vec{u}|^2|\vec{v}|^2 - (|\vec{u}| |\vec{v}| \cos \theta)^2 \\ &= |\vec{u}|^2|\vec{v}|^2(1 - (\cos \theta)^2) \\ &= |\vec{u}|^2|\vec{v}|^2(\sin \theta)^2 \end{aligned}$$

Therefore $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2(\sin \theta)^2$ which gives the desired result.

3.2 Properties

A few properties of the cross product are:

1. Skew-commutativity:

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

2. The cross product is distributive:

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

3. If $\vec{u} \times \vec{v} = \vec{0}$ then \vec{u} and \vec{v} are parallel.

Example 8. Take two vectors $\vec{u} = (1, -3, 2)$ and $\vec{v} = (2, 4, -5)$. Compute $\vec{u} \times \vec{v}$.

We compute the cross product using the formula:

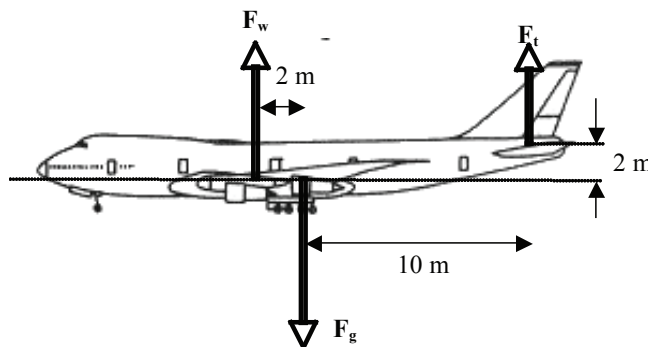
$$\begin{aligned}\vec{u} \times \vec{v} &= (-3 \cdot (-5) - 4 \cdot 2)\vec{i} + (2 \cdot (2) - (-5) \cdot 1)\vec{j} + (1 \cdot 4 - (-3) \cdot 2)\vec{k} \\ &= 7\vec{i} + 9\vec{j} + 10\vec{k}\end{aligned}$$

Example 9. Find the area of a triangle whose vertices are $A(1,0,0)$, $B(0,3,0)$ and $C(0,0,2)$.

Matlab code:

```
AB=[-1 3 0]
AB =
    -1     3     0
AC=[-1 0 2]
AC =
    -1     0     2
X=cross(AB,AC)
X =
     6     2     3
Area = norm(X)/2
Area =
    3.5000
```

Example 10. An airplane is in a steady flight as shown in the diagram. Determine the lift force produced by the tail such that there is no net torque on the aircraft. Assume $F_w = 500,000$ N.

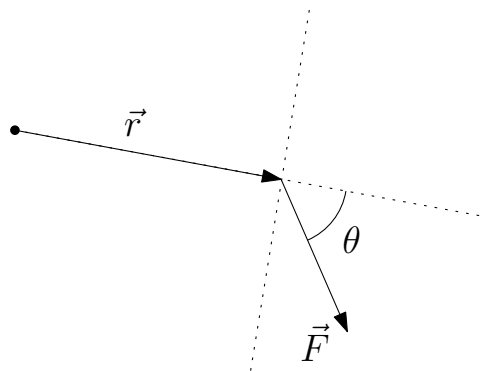


Since we assume the flight is steady, the net torque should be equal to zero. The torque due to F_w is equal to $1,000,000$. The torque due to the tail is $10F_t$. Since they have to be equal we have:

$$10 F_t = 1,000,000 \quad \rightarrow \quad F_t = 100,000 \text{ N}$$

3.3 Physical applications

Torque. We have a wrench which is applying a force \vec{F} as shown in Figure 11. What is the torque of \vec{F} ?

Figure 11: Torque due to force \vec{F}

We know that the torque is given by $|\vec{r}| |\vec{F}| \sin \theta$. This can also be expressed using the cross product as $|\vec{r} \times \vec{F}|$. The vector torque is given by $\vec{r} \times \vec{F}$. The advantage of computing the vector torque is that it gives us the direction of the torque as well as the magnitude.

The torque is important to study the rotational motion of objects. For example, in rotational equilibrium, the torque is zero. Let's assume that the force \vec{F} is in the direction of \vec{r} . Then we know that the torque is zero. This can be verified from the definition of the cross product:

$$|\vec{r} \times \vec{F}| = |\vec{r}| |\vec{F}| \sin \theta = 0 \text{ because } \theta = 0 \text{ in this case.}$$

Moving charge. We have a charge q that is moving with velocity \vec{v} in a magnetic field \vec{B} . Thus, the force on the charge is given by (see Figure 12, 13 and 14)

$$\vec{F} = (q\vec{v}) \times \vec{B}$$

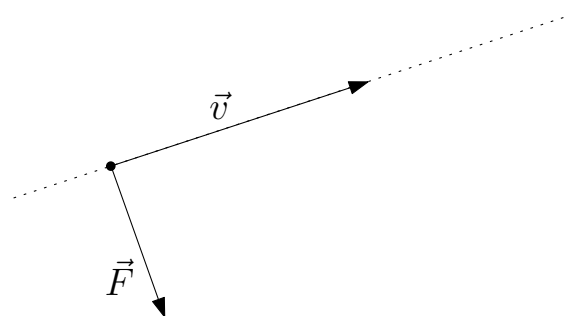


Figure 12: Force on a moving charge

From the properties of the cross product, we know that the force is therefore perpendicular to the velocity. Let's assume that the particle is moving in a straight line and has moved along a segment \vec{d} ; the velocity vector \vec{v} is in the direction of the motion and is therefore parallel to \vec{d} . Hence, we have:

$$W = \vec{F} \cdot \vec{d} = |\vec{F}| |\vec{d}| \cos \theta = 0 \text{ because } \vec{F} \text{ is perpendicular to } \vec{d}.$$

The work done is always 0. Thus, the particle cannot be accelerated using a magnetic field! One can only deflect the particle using a magnetic field, i.e., we can change the direction of the particle velocity but not its magnitude by using a magnetic field.

These properties have immense practical significance in particle accelerators where a magnetic field is used to ensure that the particle moves around in a tunnel, while an electric field accelerates the particle, as shown in Figures 13 and 14.

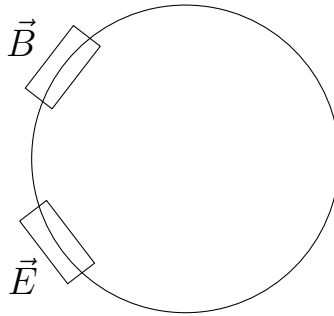


Figure 13: Schematic representation of a particle accelerator

Reading: Thomas - 12.5, 13.1, 13.2, 13.3

4 Equations for lines and planes

4.1 Lines in 2D

Let us find the equation of a line which goes through a point $P_0(x_0, y_0)$ and has the unit normal $\vec{n}(n_1, n_2)$. If the point $P(x, y)$ lies on the line (see Figure 15) then we must have $\overrightarrow{P_0P} \cdot \vec{n} = 0$, which gives the equation for the line in 2D as:

$$(x - x_0)n_1 + (y - y_0)n_2 = 0$$

4.2 Lines in 3D

The parametric representation for the equation of a 3D line turns out to be easier than an explicit equation so we will derive a parametric equation of a line in 3D. We are given the vector $\vec{v}(v_1, v_2, v_3)$ which points in the direction of the line and a point $P_0(x_0, y_0, z_0)$ lying on the line. If the point $P(x, y, z)$ lies on the line (see Figure 16), then there must exist a number t such that:

$$\overrightarrow{P_0P} = t\vec{v}$$

This is the parametric equation of a line. This **vector equation** is equivalent to three scalar equations:

$$\begin{aligned} x - x_0 &= t v_1 \\ y - y_0 &= t v_2 \\ z - z_0 &= t v_3 \end{aligned}$$

How a Synchrotron Works

4. Storage Ring

The booster ring feeds electrons into the storage ring, a many-sided donut-shaped tube. The tube is maintained under vacuum, as free as possible of air or other stray atoms that could deflect the electron beam. Computer-controlled magnets keep the beam absolutely true.

Synchrotron light is produced when the bending magnets deflect the electron beam; each set of bending magnets is connected to an experimental station or beamline. Machines filter, intensify, or otherwise manipulate the light at each beamline to get the right characteristics for experiments.

5. Focusing the Beam

Keeping the electron beam absolutely true is vital when the material you're studying is measured in billionths of a metre. This precise control is accomplished with computer-controlled quadrupole (four pole) and sextupole (six pole) magnets. Small adjustments with these magnets act to focus the electron beam.

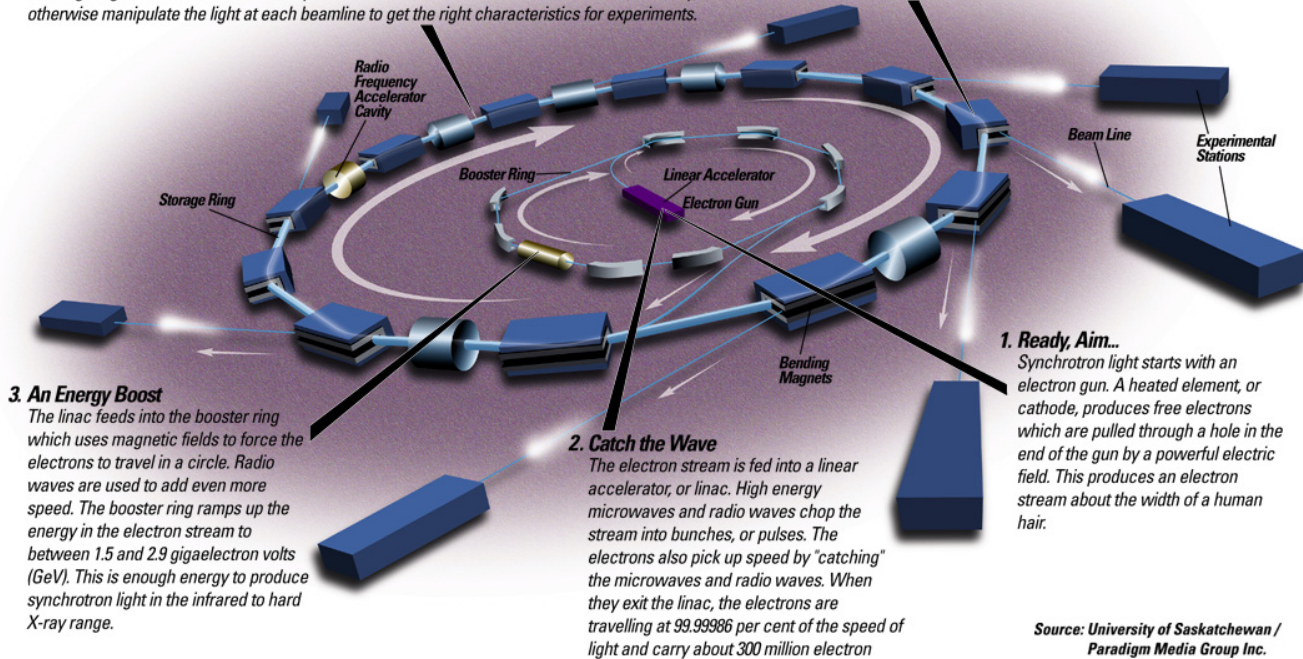


Figure 14: How a Synchrotron works

4.3 Planes in 3D

Given a point $P_0(x_0, y_0, z_0)$ lying on the plane and the unit normal $\vec{n}(n_1, n_2, n_3)$ coming out of the plane, we are going to find the equation of the plane. If the point $P(x, y, z)$ lies on the plane (see Figure 17), then clearly the vector $\overrightarrow{PP_0}$ is perpendicular to \vec{n} as $\overrightarrow{PP_0}$ lies in the plane. Thus, we have:

$$\overrightarrow{PP_0} \cdot \vec{n} = 0$$

This equation can be expanded to get:

$$(x - x_0)n_1 + (y - y_0)n_2 + (z - z_0)n_3 = 0$$

Example 11. Find an equation of the plane through $A(0, 0, 1)$, $B(2, 0, 0)$ and $C(0, 3, 0)$.

We can find a vector normal to the plane (see Figure 18) as:

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (3, 2, 6)$$

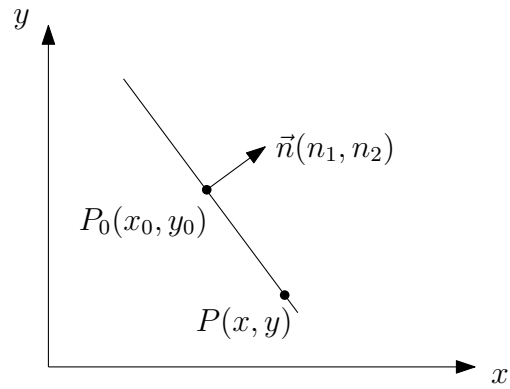


Figure 15: A line in 2D

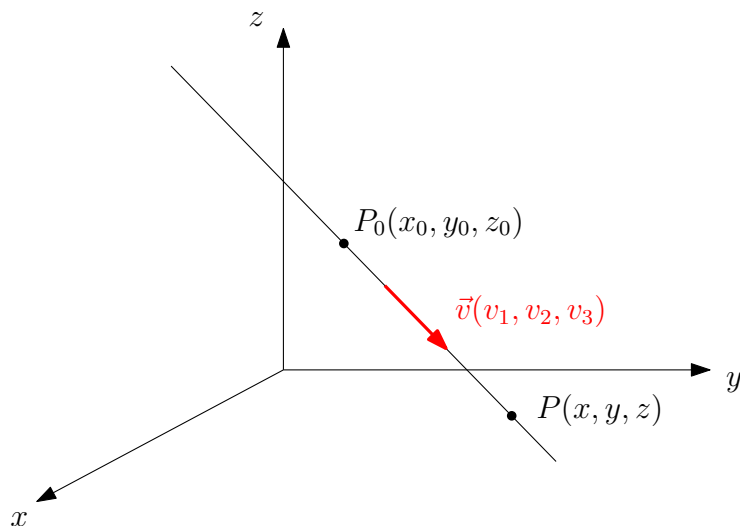


Figure 16: A line in 3D

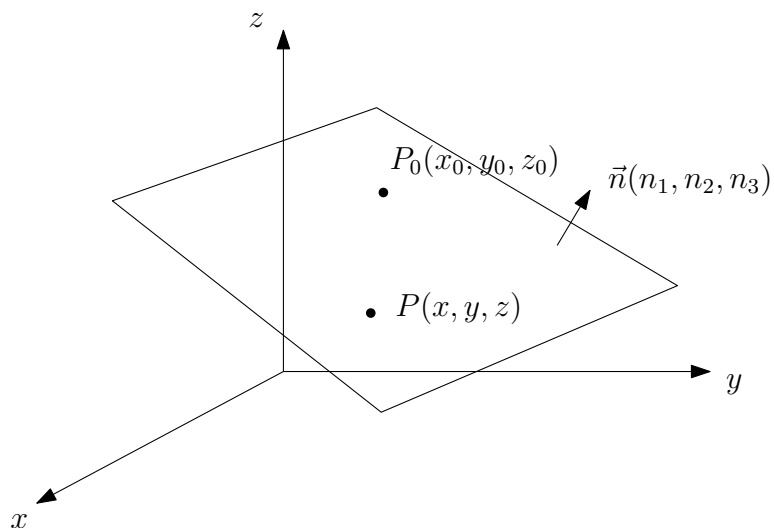
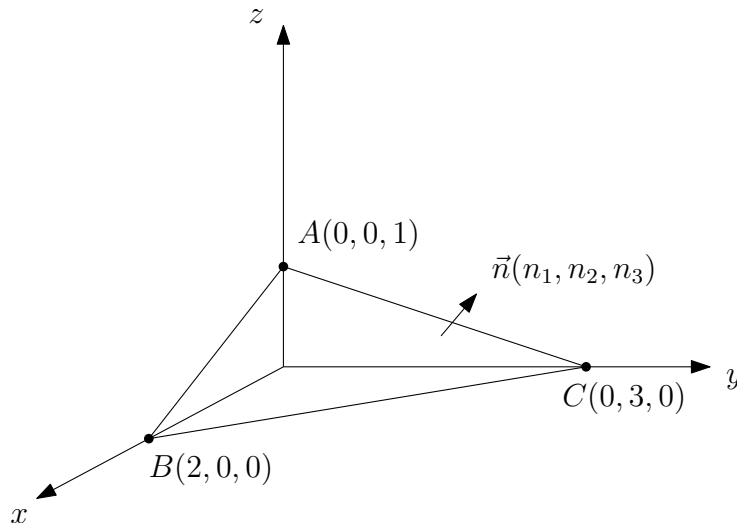


Figure 17: A plane in 3D

Figure 18: A plane through the three points A , B and C

Thus, the equation of the plane is:

$$(x - 0)3 + (y - 0)2 + (z - 1)6 = 0$$

which simplifies to

$$3x + 2y + 6z = 6$$

MATLAB code:

```
>> A = [0 0 1];
>> B = [2 0 0];
>> C = [0 3 0];
>> n = cross(B-A,C-A)
n = 3 2 6
```

Example 12. Find the equation of the line of intersection of the two planes: $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ (see Figure 19).

First, we find a point lying on the line of intersection. Since there are an infinite number of such points, we find one point on this line by taking $z = 0$ and solving

$$3x - 6y = 15$$

$$2x + y = 5$$

The solution of the above equations, $Q(3, -1, 0)$, is a point lying on the line of intersection of the planes. Now, considering the normal vectors \vec{n}_1 and \vec{n}_2 to the two planes, we can see that a vector \vec{v} in the direction of the line of intersection is perpendicular to both \vec{n}_1 and \vec{n}_2 . Thus, we can take:

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = (14, 2, 15)$$

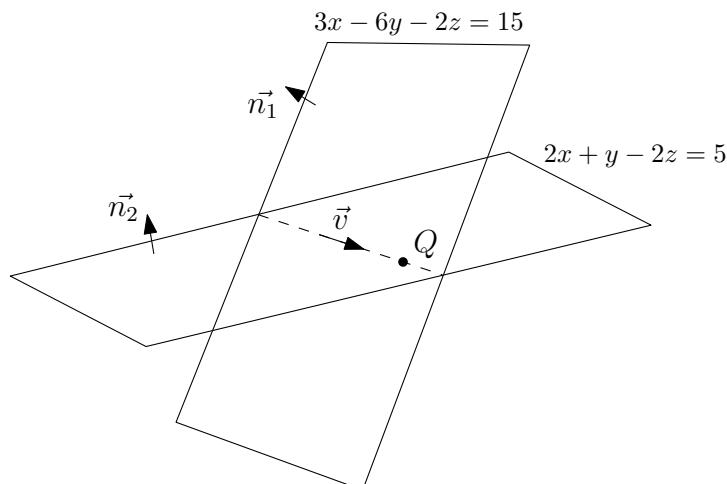


Figure 19: The line of intersection (dashed) of two planes

Thus, we can write the 3 parametric equations of the line as:

$$\begin{aligned}x &= 3 + 14t \\y &= -1 + 2t \\z &= 15t\end{aligned}$$

Example 13. Find the shortest distance from the point $Q(0, 1, 1)$ to the plane $4y + 3z = -12$ (see figure 20).

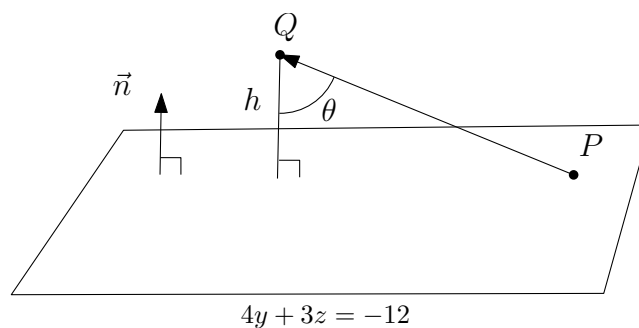


Figure 20: Shortest distance from a point to a plane

First, note that the components of the normal vector to the plane, \vec{n} , are $(0, 4, 3)$. Looking at the equation of the plane again, we can see that the point $P(0, -3, 0)$ satisfies the equation of the plane and hence lies on the plane. Take vector \vec{PQ} and draw the projection onto \vec{n} . The distance to the plane h is then given by:

$$h = |\text{proj}_{\vec{n}} \vec{PQ}| = \left| \frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|} \right| = 3.8$$

MATLAB code:

```

>> P = [0 -3 0];
>> Q = [0 1 1];
>> n = [0 4 3];
>> abs(dot(Q-P,n)/norm(n))
ans = 3.8000

```

Example 14. Find the distance (i.e., the shortest distance) from the point $P(0,0,12)$ to the line: $x = 4t$, $y = -2t$, $z = 2t$ (see Figure 21).

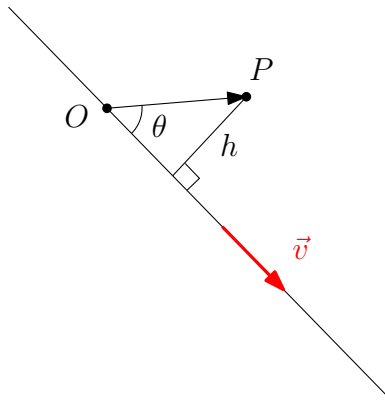


Figure 21: The distance from a point to a line

Let O be any point lying on the line and let \vec{v} be a vector in the direction of the line. From the figure, we can see that the distance h is given by:

$$h = |\overrightarrow{OP}| \sin \theta$$

We also know that:

$$|\overrightarrow{OP} \times \vec{v}| = |\overrightarrow{OP}| |\vec{v}| \sin \theta$$

Thus, we have:

$$h = \frac{|\overrightarrow{OP} \times \vec{v}|}{|\vec{v}|}$$

It is easy to check that the point $O(0,0,0)$ lies on the line and the vector \vec{v} is given by $(4,-2,2)$. Hence we have

$$h = \frac{|(24, 48, 0)|}{|(4, -2, 2)|} = \sqrt{120}$$

Example 15. Find the angle between the two planes: $x + y = 1$ and $2x + y - 2z = 2$ (see figure 22).

First, note that the angle between two planes is defined as the angle between the normal vectors of the planes. The normal to the first plane is $\vec{n}_1(1,1,0)$ and the normal to the second plane is $\vec{n}_2(2,1,-2)$. The angle between the two vectors can be computed using the dot product as

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

This can be computed using MATLAB as follows

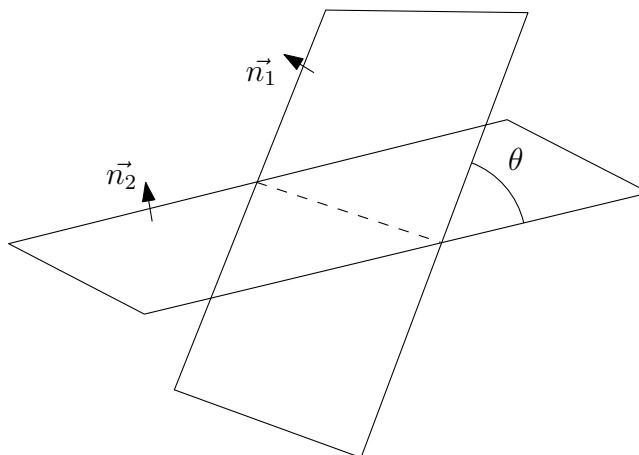


Figure 22: Angle between two planes

```

>> n1=[1 1 0]; n2=[2 1 -2];
>> cos_theta = dot(n1/norm(n1),n2/norm(n2))
cos_theta = 0.7071
>> acos(cos_theta)/pi*180
ans = 45.0000

```

Example 16. Find the distance between two skewed lines. The parametric equation of the first line is $(0, 5, -1) + t(2, 1, 3)$ and the parametric equation of the second line is $(-1, 2, 0) + t(1, -1, 0)$.

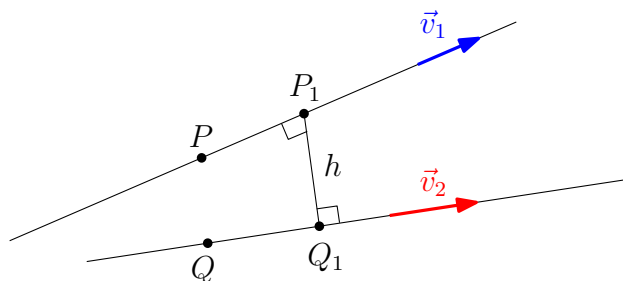


Figure 23: Distance between two skewed lines

First of all, note that we have $\vec{v}_1 = (2, 1, 3)$ and $\vec{v}_2 = (1, -1, 0)$ and also two points lying on the two lines as $P = (0, 5, -1)$ and $Q = (-1, 2, 0)$ respectively. From Figure 23, if points P_1 and Q_1 are such that $\overrightarrow{P_1Q_1}$ is perpendicular to \vec{v}_1 and \vec{v}_2 , then the distance h between P_1 and Q_1 is equal to the distance between the 2 lines. Thus, we can compute

$$\vec{v}_1 \times \vec{v}_2 = (3, 3, -3)$$

We can now project \overrightarrow{PQ} along $\vec{v}_1 \times \vec{v}_2$ to get the vector $\overrightarrow{P_1Q_1}$ and its magnitude gives the distance between the two lines as:

$$h = |\overrightarrow{P_1Q_1}| = |\text{proj}_{\vec{v}_1 \times \vec{v}_2} \overrightarrow{PQ}|$$

Thus, we have

$$h = \frac{|\overrightarrow{PQ} \cdot (\vec{v}_1 \times \vec{v}_2)|}{|\vec{v}_1 \times \vec{v}_2|}$$

Compute $\vec{v}_1 \times \vec{v}_2 = (3, 3, -3)$ and finally $h = \frac{5}{\sqrt{3}}$.

Matlab code:

```
>> P = [0 5 -1];
>> Q = [-1 2 0];
>> v1 = [2 1 3];
>> v2 = [1 -1 0];
>> abs(dot(Q-P, cross(v1, v2))/norm(cross(v1, v2)))
ans = 2.8868
```

5 Curves and paths

Path: If a particle is moving in space and we can specify the position of the particle at any time t , then the trajectory of the particle is called a path. For example:

- Line: $\vec{x}(t) = \vec{a} + t\vec{b}$
- Circle: $\vec{x}(t) = (3 \cos t, 3 \sin t)$
- Helix: $\vec{x}(t) = (a \cos t, a \sin t, bt)$

Curve: A curve is a set of points in 3D space. It can be defined as the range of a path $\vec{x}(t)$, i.e., the set of all values taken by $\vec{x}(t)$ as t is varying. For example, a line or a circle is a curve. A curve can be parameterized in different ways. For example the paths $(\cos t, \sin t)$ and $(\cos 2t, \sin 2t)$ in fact parameterize the same curve, namely the unit circle.

Figure 24 shows how MATLAB can be used to plot curves. This is an advanced example which uses color and shading.

5.1 Differentiation of vector functions – velocity and acceleration

A vector function $\vec{f}(t)$ is in general defined as:

$$\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$$

Vector functions can be differentiated like regular scalar functions:

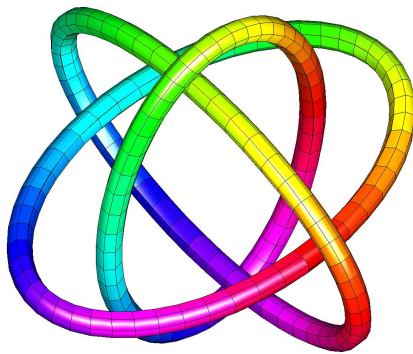
$$\frac{d\vec{f}(t)}{dt} = \frac{df_1(t)}{dt}\vec{i} + \frac{df_2(t)}{dt}\vec{j} + \frac{df_3(t)}{dt}\vec{k}$$

This has a natural application when $\vec{f}(t)$ is in fact a path $\vec{r}(t)$ representing the trajectory of a point in space:

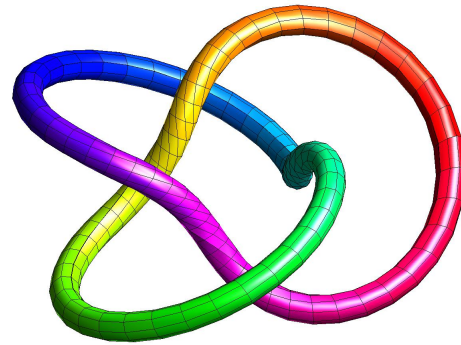
$$\vec{r}(t) = (x(t), y(t), z(t))$$

The velocity vector $\vec{v}(t)$ can be easily calculated by taking the derivative of $\vec{r}(t)$:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$



(a) The Borromean rings



(b) The trefoil knot



(c) The DNA molecule forms a double helix



(d) Collagen, a protein found in muscles and bones, forms a triple helix

Figure 24: A few curves plotted using MATLAB

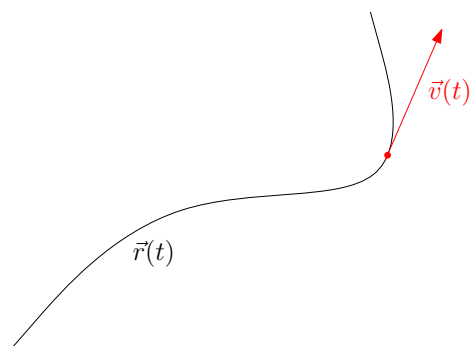


Figure 25: Velocity of a particle along a path

The velocity is always tangent to the curve parameterized by the path as shown in Figure 25. Similarly, the acceleration of the particle along the path is given by

$$\vec{a} = \frac{d\vec{v}}{dt}$$

5.2 Vector differentiation

Here are a few differentiation rules that will come in handy for differentiating vectors:

$$\begin{aligned}\frac{d}{dt} [\vec{u}(t) + \vec{v}(t)] &= \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt} \\ \frac{d}{dt} [c\vec{u}(t)] &= c \frac{d\vec{u}}{dt} \\ \frac{d}{dt} [h(t)\vec{u}(t)] &= \frac{dh}{dt} \vec{u} + h \frac{d\vec{u}}{dt} \\ \frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] &= \frac{d\vec{u}}{dt} \cdot \vec{v} + \vec{u} \cdot \frac{d\vec{v}}{dt} \\ \frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] &= \frac{d\vec{u}}{dt} \times \vec{v} + \vec{u} \times \frac{d\vec{v}}{dt} \\ \frac{d}{dt} [\vec{u}(f(t))] &= \frac{df}{dt} \frac{d\vec{u}}{df}\end{aligned}$$

Example 17. A particle is moving along the parabola $y = x^2$. The path is: $\vec{r}(t) = t\vec{i} + t^2\vec{j}$. Find the velocity, acceleration and speed at $t = 2$.

We can find the velocity and acceleration of the particle using our definitions as:

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j} \\ \vec{a} &= \frac{d\vec{v}}{dt} = 2\vec{j}\end{aligned}$$

At $t = 2$, the speed is $|\vec{v}(2)| = |\vec{i} + 4\vec{j}| = \sqrt{1 + 4^2} = \sqrt{17}$.

Example 18. A path is given by $\vec{r} = b \cos \omega t \vec{i} + b \sin \omega t \vec{j} + ct \vec{k}$. Find the velocity and acceleration.

We use our definitions again to compute

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = -b\omega \sin \omega t \vec{i} + b\omega \cos \omega t \vec{j} + c \vec{k} \\ \vec{a} &= \frac{d\vec{v}}{dt} = -b\omega^2 \cos \omega t \vec{i} - b\omega^2 \sin \omega t \vec{j}\end{aligned}$$

Note that the path is a helix which translates in the \vec{k} direction; the computed acceleration always points towards the center of the circle that the helix moves in. This can be seen in Figure 26.

Application of this example: Assume we have a charged particle with charge q , moving in a magnetic field \vec{B} along the path defined above. We know that the force on the particle is given by $\vec{F} = (q\vec{v}) \times \vec{B}$, where $\vec{B} = B\vec{k}$. Thus, the force is equal to:

$$\vec{F} = qb\omega B(\cos \omega t \vec{i} + \sin \omega t \vec{j})$$

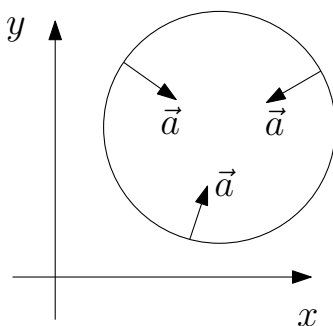


Figure 26: Acceleration of a helical path (top view)

Also, from Newton's second law we have $\vec{F} = m\vec{a}$. Equating these two, we get:

$$-mb\omega^2 = qb\omega B$$

This simplifies to:

$$\omega = -\frac{qB}{m}$$

This tells us that **the angular velocity of the particle is determined by intensity of the magnetic field \vec{B} .**

5.3 Vector integration

Any vector function $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ can be integrated and:

$$\int_a^b \vec{f}(t) dt = \left(\int_a^b f_1(t) dt \right) \vec{i} + \left(\int_a^b f_2(t) dt \right) \vec{j} + \left(\int_a^b f_3(t) dt \right) \vec{k}$$

For example, we can integrate the velocity vector $\vec{v}(t)$ to get the trajectory $\vec{r}(t)$:

$$\vec{r}(t) - \vec{r}(0) = \int_0^t \vec{v}(\tau) d\tau$$

5.4 Application to projectile motion

Example 19. A projectile is launched from the origin at time $t = 0$ with an initial velocity v_0 at an angle θ with respect to the original (see figure 27).

1. Determine the maximum height that the projectile reaches.
2. Determine the flight time and range.

The only force on the projectile is due to gravity which is acting vertically down and hence Newton's second law gives us $\vec{F} = m\vec{a} = -mg\vec{j}$, which in turn says that the acceleration is given by $\vec{a} = -g\vec{j}$. We can now integrate the expression for acceleration to get the velocity as:

$$\vec{v}(t) - \vec{v}_0 = \int_0^t \vec{a}(\tau) d\tau = -gt\vec{j}$$

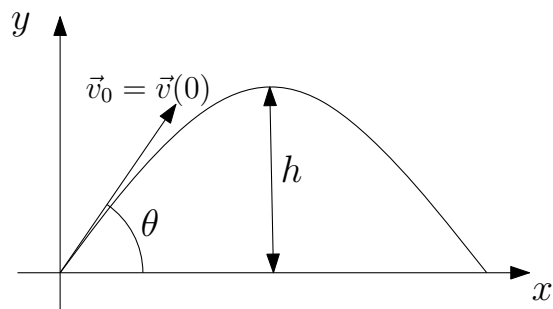


Figure 27: The motion of a projectile

The initial velocity \vec{v}_0 is given by $v_0(\cos \theta \vec{i} + \sin \theta \vec{j})$. Using this initial condition, we get

$$\vec{v}(t) = v_0 \cos \theta \vec{i} + (v_0 \sin \theta - gt) \vec{j}$$

We can again integrate to get the trajectory of the projectile as:

$$\begin{aligned} \vec{r}(t) &= \int_0^t \vec{v}(\tau) d\tau \\ &= v_0(\cos \theta)t \vec{i} + \left(v_0(\sin \theta)t - \frac{gt^2}{2} \right) \vec{j} \end{aligned}$$

Now, note that the projectile reaches the maximum height when the \vec{j} component of the velocity is zero, i.e., at $t = v_0 \sin \theta / g$. We can use this time to compute the maximum height h as

$$h = v_0 \sin \theta \left(\frac{v_0 \sin \theta}{g} \right) - \frac{g}{2} \left(\frac{v_0 \sin \theta}{g} \right)^2 = \frac{v_0^2 \sin^2 \theta}{2g}$$

To compute the flight time, note that the flight ends when the projectile hits the ground, i.e., when the \vec{j} component of \vec{r} is 0, which gives us the condition:

$$v_0(\sin \theta)t_F - \frac{gt_F^2}{2} = 0$$

Thus, we have the flight time t_F as:

$$t_F = \frac{2v_0 \sin \theta}{g}$$

Finally, now that we know the flight time, we can use it to easily compute the range by computing the horizontal (that is along \vec{i}) component of $\vec{r}(t)$ when t is equal to the flight time:

$$\text{Range} = v_0 \cos \theta \left(\frac{2v_0 \sin \theta}{g} \right) = \frac{v_0^2 \sin 2\theta}{g}$$

where we have used the trigonometric identity: $2 \sin \theta \cos \theta = \sin(2\theta)$.

This expression for the range tells us that we get the maximum range for $\theta = 45^\circ$ as $\sin(2\theta)$ attains its maximum value, i.e., 1, for $\theta = 45^\circ$ and hence the maximum possible range for any θ is given by v_0^2/g .

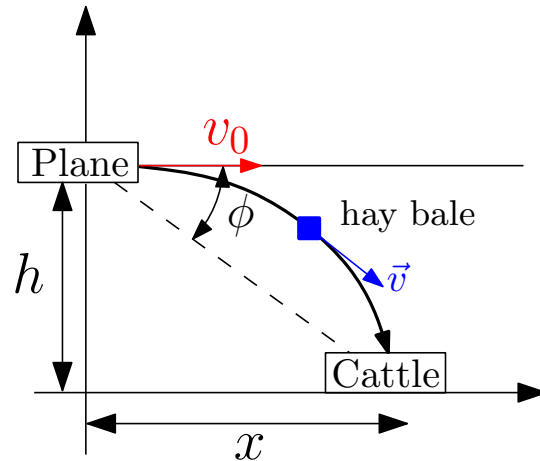


Figure 28: Snow bound cattle

Example 20. *Snow bound cattle.* A plane is flying at an altitude of 1600 ft at a speed of 220 ft/sec. What is the angle of sight at which the plane should release hay bales for snow bound cattle (see figure 28)?

We will denote the height of the plane as $h = 1600$ ft and the horizontal velocity of the plane as $v_0 = 220$ ft/sec. Also, let the horizontal distance between the plane and the cattle be x ft. Then from the figure, we have $\tan \phi = \frac{h}{x}$. Also, the only force acting on the hay bales is gravity and hence the acceleration of the hay bales is given by $\vec{a} = -g\vec{j}$ which can be integrated w.r.t. time to get the velocity of the hay bales as

$$\vec{v} = v_0\vec{i} - gt\vec{j}$$

Integrate this again to get the trajectory of the hay bales as:

$$\vec{r} = v_0t\vec{i} + h\vec{j} - \frac{gt^2}{2}\vec{j}$$

The hay bales hit the ground when $h - \frac{gt^2}{2} = 0$, i.e., when $t = \sqrt{\frac{2h}{g}}$. Thus, we have:

$$x = v_0\sqrt{\frac{2h}{g}}$$

and consequently, the angle is given by

$$\tan \phi = \frac{1}{v_0}\sqrt{\frac{gh}{2}}$$

MATLAB code

```
>> g=9.81; % m/s
>> conv=0.3048; % To change the units to metric units
>> v0=220*conv % metric units
v0 =
    67.0560
```

```

>> h=1600*conv % metric units
h =
    487.6800
>> tan_phi=1/v0*sqrt(h*g/2)
tan_phi =
    0.7294
>> phi=atan(tan_phi)/pi*180 % deg
phi =
    36.1060

```

6 Differential geometry

Differential geometry deals with application of calculus to geometry, e.g., calculus and analysis are used to understand the geometry of curves and surfaces.

6.1 Length of a curve

Assume that we have a parametrization of a curve $\vec{r}(t)$. Let us say that we are at a location $\vec{r}(t)$ at time t and move to the location $\vec{r}(t + \Delta t)$ at the time $t + \Delta t$. Then we will approximate the arc length with the straight line joining the two points as shown in figure 29:

$$\Delta L = |\vec{r}(t + \Delta t) - \vec{r}(t)|$$

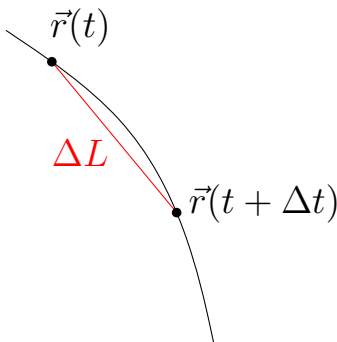


Figure 29: Computing the arc length by approximation

Also, from basic calculus we have:

$$\vec{r}(t + \Delta t) = \vec{r}(t) + \Delta t \vec{v}(t) + \dots$$

Thus, the speed along the path is given by:

$$\frac{dL}{dt} \approx \frac{\Delta L}{\Delta t} = |\vec{v}(t)|$$

And consequently the length of the path is given by

$$L = \int_{t=0}^T \frac{dL}{d\tau} d\tau = \int_{t=0}^T |\vec{v}(\tau)| d\tau$$

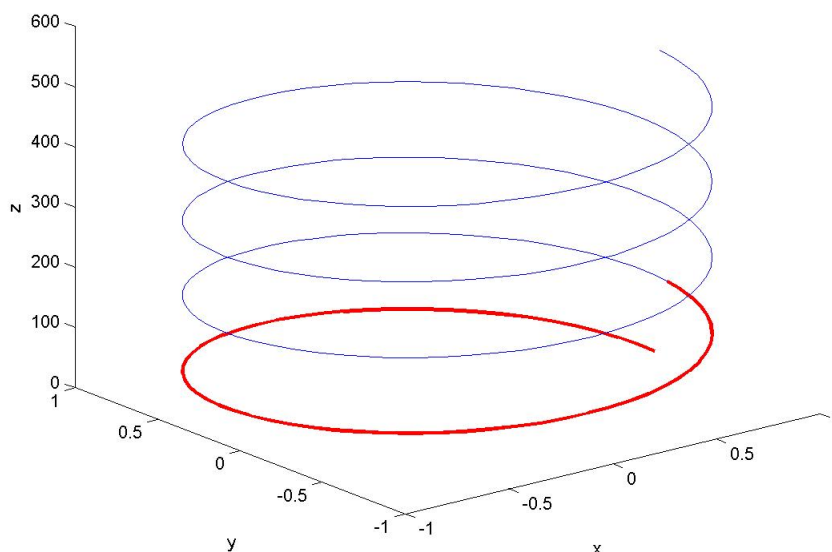


Figure 30: A helix, with one turn shown in red

Example 21. Find the arc length for one turn of $\vec{r} = b \cos \omega t \vec{i} + b \sin \omega t \vec{j} + ct \vec{k}$.

The given path describes a helix (see Figure 30) whose axis points in the positive z direction. We can compute the velocity along the path by differentiating the position vector as follows:

$$\vec{v}(t) = -b\omega \sin \omega t \vec{i} + b\omega \cos \omega t \vec{j} + c \vec{k}$$

This gives the magnitude of the velocity (speed) as

$$|\vec{v}(t)| = \sqrt{(b\omega)^2 \sin^2 \omega t + (b\omega)^2 \cos^2 \omega t + c^2} = \sqrt{(b\omega)^2 + c^2}$$

Thus the arc length of one turn is given by integrating the speed as:

$$L = \int_0^{\frac{2\pi}{\omega}} |\vec{v}(\tau)| d\tau = \frac{2\pi}{\omega} \sqrt{(b\omega)^2 + c^2}$$

6.2 Arc length function

Given a parametrization of a curve $\vec{r}(t)$, we define the arc length function $s(t)$ as follows (as shown in Figure 31):

$$s(t) = \int_0^t |\vec{v}(\tau)| d\tau$$

The arc length function $s(t)$ is used to re-parameterize the path ($\vec{r}(t) \rightarrow \vec{r}(t(s))$) in such a way that the new parametrization depends solely on the geometry of the curve. For example, using the parametrization by s , we get:

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \frac{dt}{ds} = \frac{d\vec{r}}{dt} \frac{1}{(ds/dt)} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

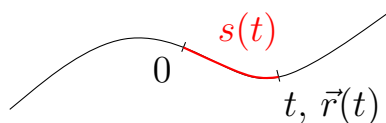


Figure 31: The arc length function

The above expression depends solely on the geometry of the curve and not how fast the curve is traced by $\vec{r}(t)$. In fact $\vec{r}(s)$ has unit speed, i.e., $s(t)$ is such that the particle is tracing the curve with unit speed.

Example 22. Re-parameterize the helix $\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$ with respect to arc length measured from $(1, 0, 0)$.

We can calculate $\frac{ds}{dt} = \left| \frac{d\vec{r}(t)}{dt} \right| = \sqrt{2}$, and so:

$$s = s(t) = \int_0^t \left| \frac{d\vec{r}(\tau)}{d\tau} \right| d\tau = (\sqrt{2}) t$$

Therefore $t = \frac{s}{\sqrt{2}}$ and the re-parametrization is obtained by substituting for t :

$$\vec{r}(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right) \vec{i} + \sin\left(\frac{s}{\sqrt{2}}\right) \vec{j} + \left(\frac{s}{\sqrt{2}}\right) \vec{k}$$

You can check that with this re-parametrization, the speed of $\vec{r}(t(s))$ is 1.

6.3 Unit tangent vector

The tangent vector to a curve is defined as (see Figure 32):

$$\vec{T} = \frac{d\vec{r}}{ds}$$

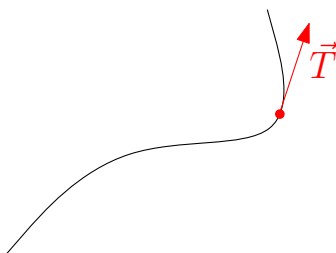


Figure 32: The tangent vector

This is a unit vector and it is parallel to the velocity vector \vec{v} . We can compute \vec{T} using the above definition in terms of the velocity vector as:

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \frac{dt}{ds} = \vec{v}(t) \frac{1}{ds/dt} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

6.4 Curvature

Intuitively, we can think of curvature as measuring the amount of “curviness” near a point. For example, the curvature K of a straight line is 0 as the curve is straight; the curvature of a circle is given by the reciprocal of the radius $K = 1/r$.

Formally, the curvature K is defined as the rate with which the direction of the unit tangent vector \vec{T} is changing, i.e.:

$$K = \left| \frac{d\vec{T}}{ds} \right|$$

We can compute this by applying the chain rule to evaluate the right hand side as:

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \frac{dt}{ds} = \frac{d\vec{T}}{dt} \frac{1}{\frac{ds}{dt}} = \frac{1}{|\vec{v}|} \frac{d\vec{T}}{dt}$$

Thus, we get the following expression for the curvature:

$$K = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

This is easy to compute since curves are usually described using a parametrization $\vec{r}(t)$ from which $\vec{T}(t)$ and $\vec{v}(t)$ can be easily calculated.

Example 23. For the trajectory $\vec{r} = b \cos t \vec{i} + b \sin t \vec{j} + ct \vec{k}$, determine the following: velocity and acceleration, unit tangent vector, and curvature.

Velocity and acceleration vectors: The velocity vector can be obtained by differentiating the path w.r.t. time:

$$\vec{v} = -b \sin t \vec{i} + b \cos t \vec{j} + c \vec{k}$$

Similarly, the acceleration vector can be obtained by differentiating the velocity w.r.t. time:

$$\vec{a} = -b \cos t \vec{i} - b \sin t \vec{j}$$

Unit tangent vector: Note that $|\vec{v}| = \sqrt{b^2 + c^2}$. We can thus compute the tangent vector as:

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = -\frac{b}{\sqrt{b^2 + c^2}} \sin t \vec{i} + \frac{b}{\sqrt{b^2 + c^2}} \cos t \vec{j} + \frac{c}{\sqrt{b^2 + c^2}} \vec{k}$$

Curvature: The curvature is defined as:

$$K = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

We can differentiate the unit tangent vector to get:

$$\frac{d\vec{T}}{dt} = -\frac{b}{\sqrt{b^2 + c^2}} \cos t \vec{i} - \frac{b}{\sqrt{b^2 + c^2}} \sin t \vec{j}$$

Thus, we have

$$\left| \frac{d\vec{T}}{dt} \right| = \frac{b}{\sqrt{b^2 + c^2}}$$

Now we use this result to get the curvature as

$$K = \frac{b}{b^2 + c^2}$$

Reading: Thomas - 13.4, 13.5, 13.6, 14.1, 14.2, 14.3, 14.4

7 Moving frame and torsion

7.1 TNB frame

We are now going to define the bi-normal vector \vec{N} . For this we need to calculate the derivative of the unit tangent vector \vec{T} with respect to s . Let's first establish that $\frac{d\vec{T}}{ds}$ is perpendicular to \vec{T} .

$$\begin{aligned} |\vec{T}(s)|^2 &= 1, & (\vec{T} \text{ has unit length}) \\ \vec{T}(s) \cdot \vec{T}(s) &= 1; & \text{Differentiate both sides:} \\ \frac{d}{dt}(\vec{T}(s) \cdot \vec{T}(s)) &= 0 \\ \vec{T} \cdot \frac{d\vec{T}}{ds} + \frac{d\vec{T}}{ds} \cdot \vec{T} &= 0; & \text{Therefore:} \\ 2\vec{T} \cdot \frac{d\vec{T}}{ds} &= 0 \end{aligned}$$

We have that \vec{T} is always perpendicular to $\frac{d\vec{T}}{ds}$. We can now define the **Principal Normal Vector** as follows:

$$\vec{N} = \frac{\frac{d\vec{T}}{ds}}{\left| \frac{d\vec{T}}{ds} \right|} = \frac{1}{K} \cdot \frac{d\vec{T}}{ds}$$

\vec{N} is always perpendicular to \vec{T} due to the above property. Also, these two vectors \vec{T} and \vec{N} define a plane called the osculating plane an example of which for a helix is shown in Figure 33.

The bi-normal vector \vec{B} is defined using the tangent vector and the normal vector as

$$\vec{B} = \vec{T} \times \vec{N}$$

This trio of vectors \vec{T} , \vec{N} and \vec{B} defines a right-handed frame (coordinate system) called the TNB frame.

A crucial difference between the TNB frame and the regular xyz frame is that the TNB frame is not stationary. In fact it follows the particle as it moves along the curve, as shown in Figure 34.

7.2 Torsion

It can be shown from the above definition of \vec{B} , that $\frac{d\vec{B}}{ds}$ is parallel to \vec{N} . We can then define the **torsion** τ as the coefficient of proportionality:

$$\frac{d\vec{B}}{ds} = -\tau \vec{N}(s)$$

We can rewrite this equation in terms of t to get the equation for torsion as

$$\tau = -\frac{1}{|\vec{v}|} \left(\frac{d\vec{B}}{dt} \cdot \vec{N} \right)$$

Intuitively, the torsion measures by how much the path twists out of the osculating plane. You can visualize this on Figure 34.

Example 24. For the vector valued function $\vec{r} = b \cos t \vec{i} + b \sin t \vec{j} + ct \vec{k}$, determine the unit tangent vector, the curvature, the unit normal vector, the bi-normal vector, and the torsion.

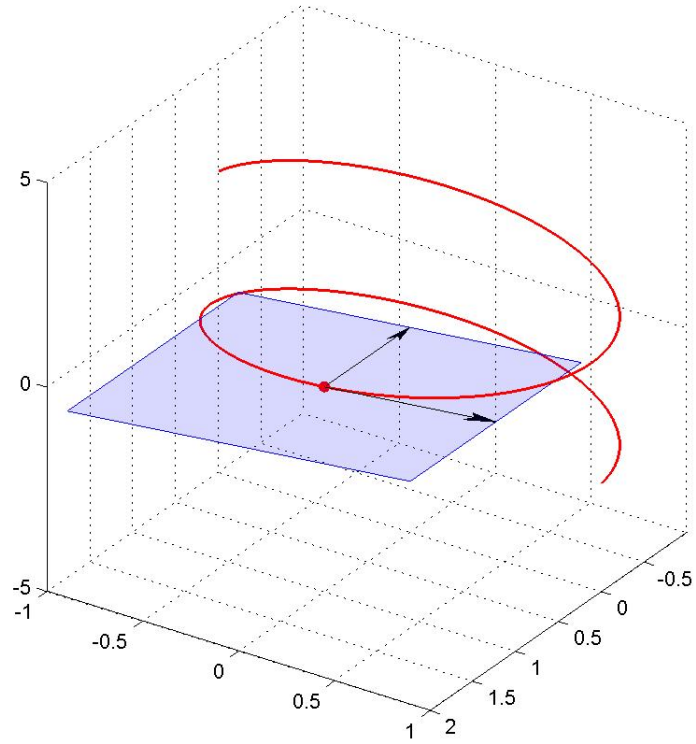


Figure 33: The osculating plane for a helix

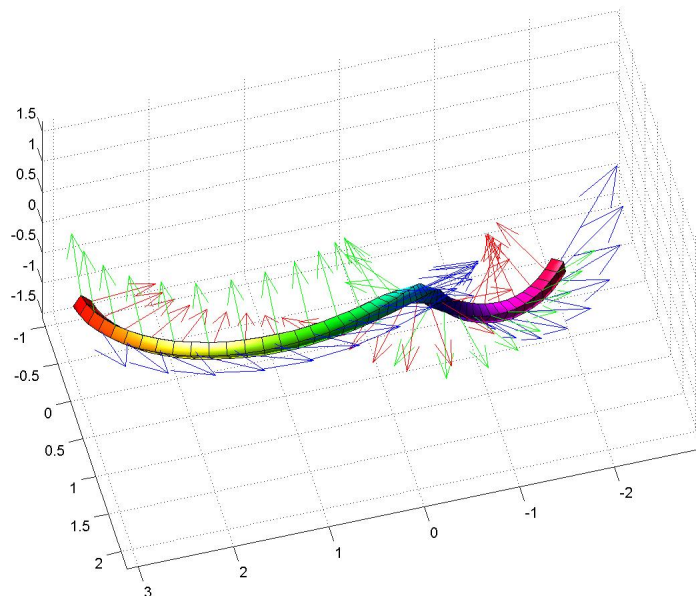


Figure 34: The TNB frame at different points for a knot

- Unit normal vector: Recall from the previous lecture that we computed the following quantities:

– The unit tangent vector:

$$\vec{T} = -\frac{b}{\sqrt{b^2 + c^2}} \sin t \vec{i} + \frac{b}{\sqrt{b^2 + c^2}} \cos t \vec{j} + \frac{c}{\sqrt{b^2 + c^2}} \vec{k}$$

– The derivative of the unit tangent vector:

$$\left| \frac{d\vec{T}}{dt} \right| = \frac{b}{\sqrt{b^2 + c^2}}$$

– The curvature

$$K = \frac{b}{b^2 + c^2}$$

We can now find the unit normal vector using our definition as

$$\vec{N}(t) = -\cos t \vec{i} - \sin t \vec{j}$$

- Torsion: We can compute the bi-normal vector as

$$\begin{aligned} \vec{B} &= \vec{T} \times \vec{N} \\ &= \frac{1}{\sqrt{b^2 + c^2}} \left(c \sin t \vec{i} - c \cos t \vec{j} + b \vec{k} \right) \end{aligned}$$

Differentiate this to get

$$\frac{d\vec{B}}{dt} = \frac{c}{\sqrt{b^2 + c^2}} (\cos t \vec{i} + \sin t \vec{j})$$

Now, compute

$$\frac{d\vec{B}}{dt} \cdot \vec{N} = -\frac{c}{\sqrt{b^2 + c^2}}$$

Recall from the previous lecture that we have $|\vec{v}| = \sqrt{b^2 + c^2}$. Thus, we can compute the torsion τ as

$$\tau = -\frac{1}{\vec{v}} \left(\frac{d\vec{B}}{dt} \cdot \vec{N} \right) = \frac{c}{b^2 + c^2}$$

7.3 Additional formulas

Here are additional formulas to compute previously defined quantities:

- The curvature:

$$K = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

- The torsion (if you know how to compute the determinant of a 3×3 matrix):

$$\tau = \frac{1}{|\vec{v} \times \vec{a}|^2} \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

7.4 Tangential and normal components of velocity and acceleration

The velocity can be expressed as

$$\vec{v} = |\vec{v}| \vec{T} = v \vec{T}$$

The acceleration is defined as

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\vec{T}) = \frac{dv}{dt} \vec{T} + v \frac{d\vec{T}}{dt}$$

Using the fact that the curvature $K = \frac{1}{v} \left| \frac{d\vec{T}}{dt} \right|$ and the normal vector satisfies

$$\vec{N} = \frac{1}{\left| \frac{d\vec{T}}{dt} \right|} \frac{d\vec{T}}{dt} = \frac{1}{Kv} \frac{d\vec{T}}{dt},$$

we have:

$$\vec{a}(t) = \frac{dv}{dt} \vec{T} + Kv^2 \vec{N}$$

The acceleration has a very simple expression in the TNB frame!

It's even possible to avoid computing the curvature with the following trick. Let the tangential component of the acceleration be a_T and the normal component be a_N . Thus, we can write $\vec{a} = a_T \vec{T} + a_N \vec{N}$. Using the pythagorean theorem (as $\vec{T} \perp \vec{N}$) we can see that

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$$

a_N can therefore be computed directly from a and a_T . This is useful in some problems.

Application: Design of train tracks. We want to design a train track to join two existing tracks smoothly (see Figure 35).

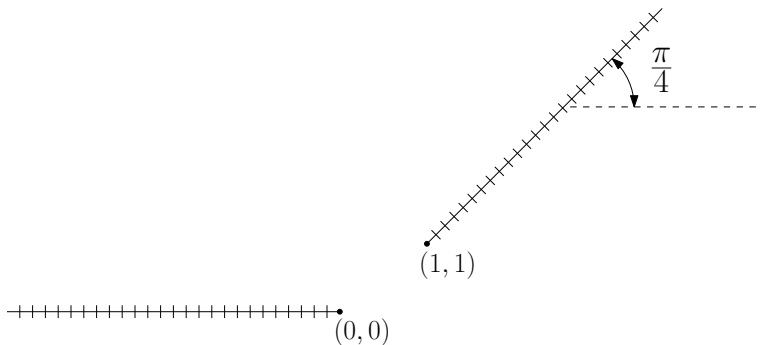


Figure 35: The train tracks

Joining the tracks with a straight line will not work as we want it to be smooth, i.e., the slope and *curvature* should be continuous. Assume for simplicity that we look for a polynomial representation of the tracks:

$$y(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$$

For the curvature and slope to be continuous at $(0,0)$ and $(1,1)$, we need the following conditions to be satisfied: $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ and $y(1) = 1$, $y'(1) = 1$ and $y''(1) = 0$. The second derivative conditions are due to the curvature. We can impose these conditions on our polynomial function and solve for the 6 variables from the 6 equations that we get. The resulting curve is depicted in Figure 36 and the resulting equation is:

$$y(x) = 3x^5 - 8x^4 + 6x^3$$

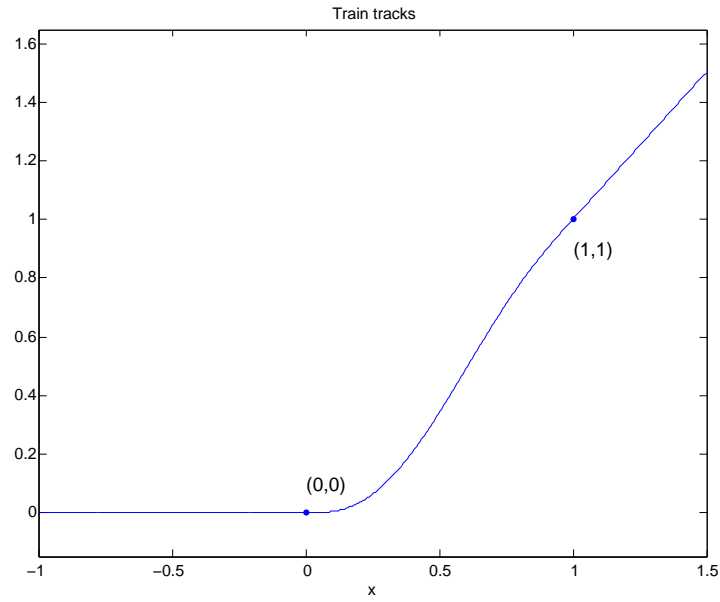


Figure 36: The connected train tracks

8 Kepler's laws

Kepler gave 3 laws for planetary motion which were based on his experimental observations. It is interesting to note that these laws were not based on physics laws but were an attempt to understand observations that Kepler had made during his study of planets. Later on, scientists like Newton and Bernoulli proved these laws mathematically. Kepler's laws are as follows:

1. The orbit of a planet is elliptical, with the sun at one of the foci of this ellipse. An ellipse is shown in Figure 37.

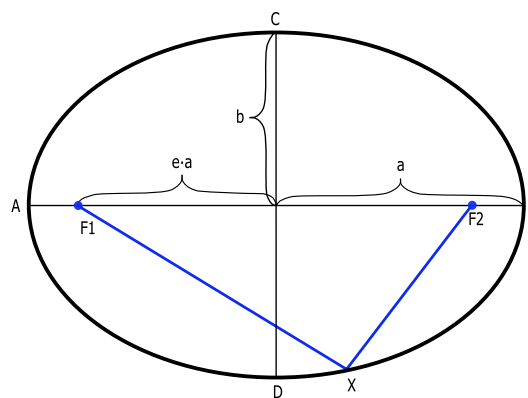


Figure 37: An ellipse with the two foci F_1 and F_2 shown as blue dots and the length of semi-major and semi-minor axis as a and b respectively.

2. During equal periods of time, a planet sweeps through equal areas with respect to the sun. This is illustrated in Figure 38.

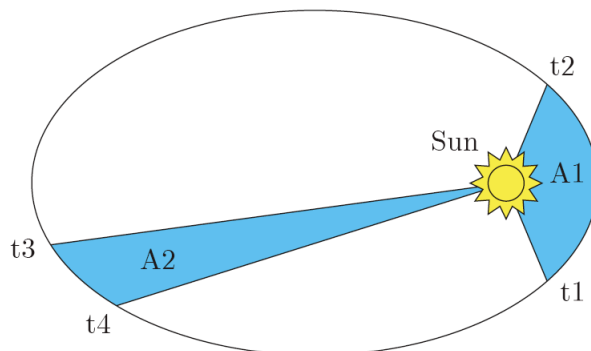


Figure 38: Kepler's second law states that if $t_2 - t_1 = t_4 - t_3$, then $A_1 = A_2$, where A_1 and A_2 are the areas of the shaded regions.

3. Let T be the period of the revolution and a be the semi-major axis of the ellipse. Then we have

$$T^2 = \frac{4\pi^2}{GM} a^3$$

where $G = 6.67 \times 10^{11} \text{ Nm}^2\text{Kg}^{-2}$ and M is the mass of the object around which the planet is orbiting, e.g., the sun for which $M = 1.98892 \times 10^{30} \text{ Kg}$.

In case you are interested, the proof of correctness of Kepler's laws is in the textbook.

Example 25. A communications relay satellite is placed in a circular orbit around the Earth. Its period of revolution is 24 hours. Given that the moon period is 27.32 days and that its circular orbit is 238,850 miles, what is that semi-major axis a of the satellite?

Using Kepler's third law, we have that $T^2 \propto a^3$. Thus, we must have

$$\frac{T_{\text{moon}}^2}{a_{\text{moon}}^3} = \frac{T_{\text{sat}}^2}{a_{\text{sat}}^3}$$

Thus, we can compute

$$a_{\text{sat}} = a_{\text{moon}} \left(\frac{T_{\text{sat}}}{T_{\text{moon}}} \right)^{2/3} = 26,330 \text{ miles}$$

9 Functions of several variables

Domain: The domain of a function is the largest set for which the function is defined.

Example 26. Function $f = \sqrt{15 - x^2 - y^2}$.

Domain: disc of radius 5.

Range: For a function $f : X \rightarrow Y$, the range is the set of all elements of Y that are actual values of f .

Example 27. Function $f = \sqrt{x^2 + y^2 + z^2}$.

Range is $[0, \infty)$.

The square parenthesis indicates that the end point is included in the interval, and round parenthesis means that the end point is not included in the interval.

9.1 Visualizing functions

Some common methods of visualizing functions are the following:

1. **Graph** of the function $f(x, y)$: this is the set of all points $(x, y, f(x, y))$. An example of a graph is shown in Figure 39.

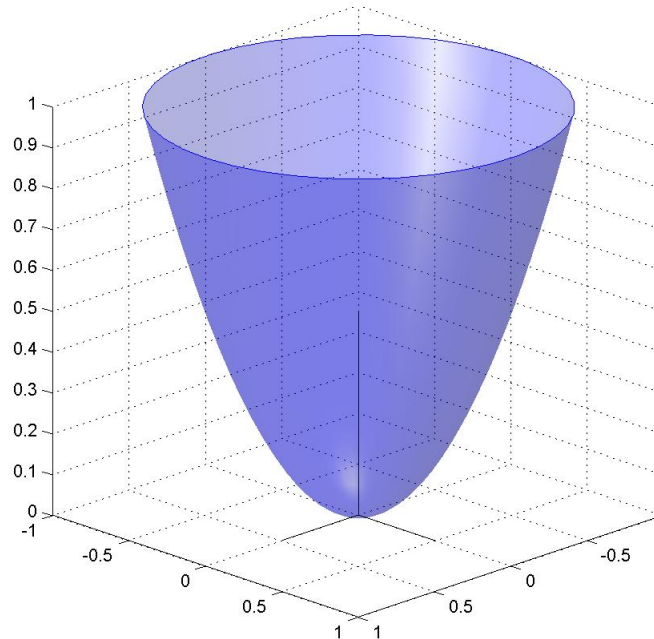


Figure 39: Graph of the function $f(x, y) = x^2 + y^2$ created using MATLAB

2. **Contour curves** are defined as the locus of points (x, y, z) (a curve in 3D) such that $f(x, y) = z = c$. They are obtained by intersecting the graph with planes of constant height: $z = c$. An example of the contour curves is shown in Figure 40.
3. **Level curves** are the projection of contour lines in the x - y plane. For example, the level curves for the function $f(x, y) = y^2 - x^2$ are plotted in Figure 41. Another application of level curves is topographic maps, which are used very frequently to generate terrain maps, an example of which is the lonely mountain map shown in Figure 42.
4. **Level surfaces:** To visualize a function of 3 variables, for example $f(x, y, z) = x^2 + y^2 + z^2$, we can plot its level surfaces, i.e., surfaces for which f is constant. Here, $x^2 + y^2 + z^2 = c$ is a level surface of this function, in this case a sphere centered at the origin.

9.2 Limits and Continuity

9.2.1 Limit

Recall that in one dimension, we have

$$\lim_{x \rightarrow a} f(x) = L$$

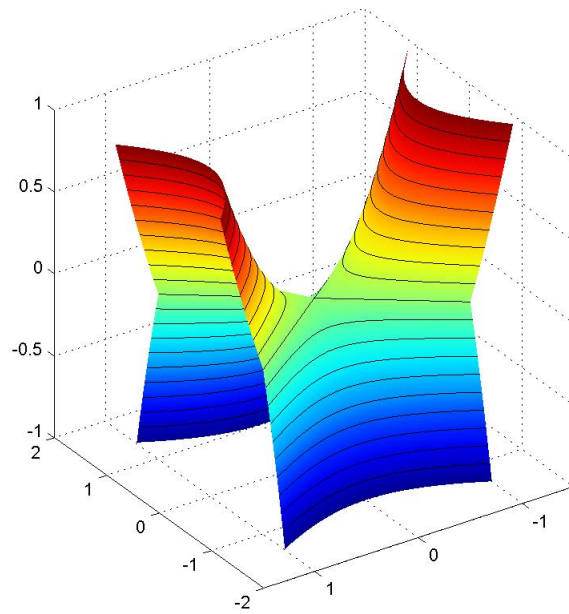


Figure 40: Contour curves for the function $f(x, y) = y^2 - x^2$ (plotted using MATLAB)

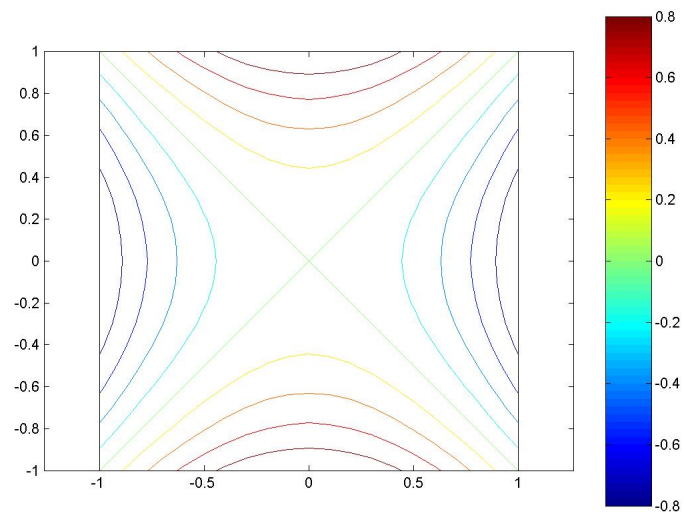


Figure 41: Level curves of the function $f(x, y) = y^2 - x^2$ (plotted using MATLAB)

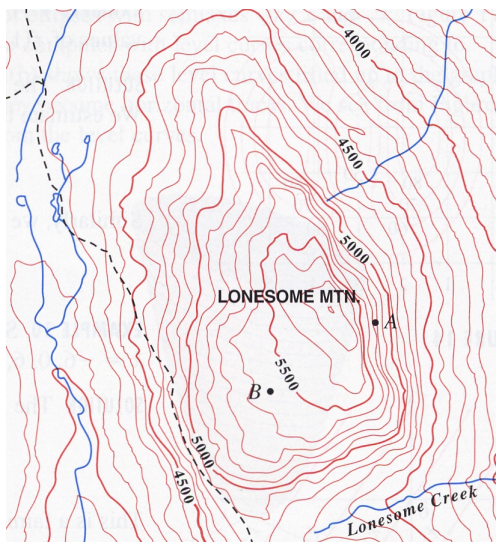


Figure 42: A typical topographic map of a mountain

This can be generalized in two dimensions to

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$$

Definition: L is the limit of f at (x_0, y_0) if one can make the value of f arbitrarily close to L by keeping (x, y) close to (x_0, y_0) .

Example 28. Find the limit: $\lim_{(x,y) \rightarrow (2,3)} xy$.

Answer:

$$\lim_{(x,y) \rightarrow (2,3)} xy = 2 \cdot 3 = 6$$

Example 29. Find the limit:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

The **Taylor expansion** of a function $f(x)$ is defined as:

$$f(x) = f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \dots$$

Use the Taylor expansion of $\sin(t)$:

$$\sin(t) = t - \frac{t^3}{6} + \dots$$

Thus, we have

$$\frac{\sin(t)}{t} = 1 - \frac{t^2}{6} + \dots$$

Using this fact, we have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$$

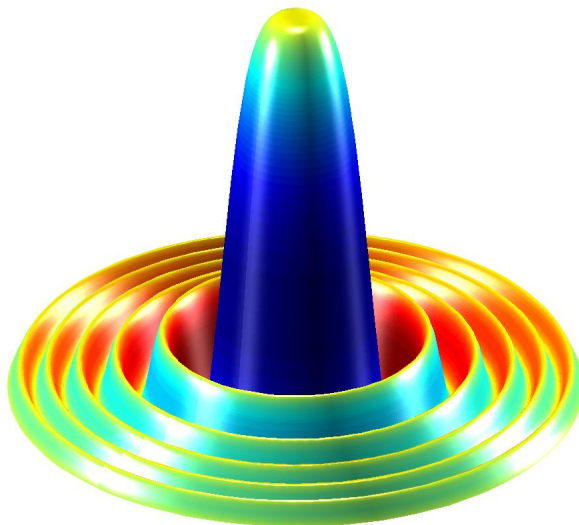


Figure 43: Plot of the function $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$

We can also find this answer using l'Hôpital's rule. The function $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ is plotted in Figure 43.

Example 30. Does the following limit exist:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} ?$$

Let's examine the following cases:

1. Approaching $(0,0)$ along $y = 0$: we know that $f(x, y) = 0$ along the x -axis and hence the limit along this line is 0.
2. Approaching $(0,0)$ along $x = 0$: we know that $f(x, y) = 0$ along the y -axis and hence the limit along this line is also 0.
3. In general, approaching $(0,0)$ along $y = mx$, our function is equal to:

$$f = \frac{x(mx)}{x^2 + (mx)^2} = \frac{m}{1 + m^2}$$

Thus, depending on the direction of approach, f converges to different values which implies that **the function has no limit at $(0,0)$.**

The above computations can be visualized on the function plot shown in Figure 44.

9.2.2 Continuity

The function f is continuous at point P if:

1. f is defined at P .

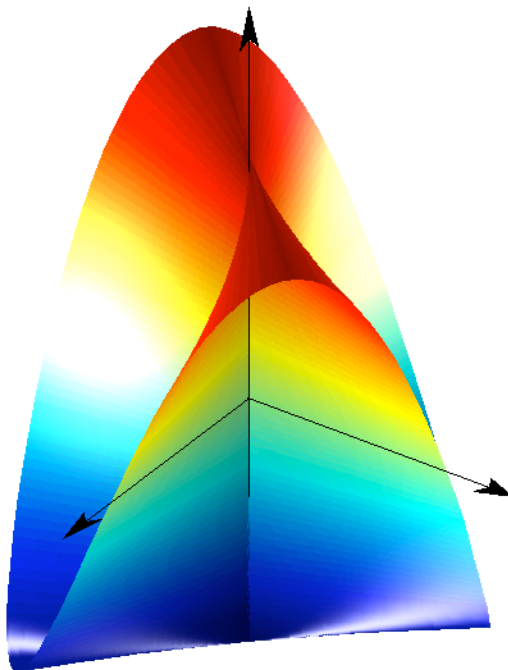


Figure 44: MATLAB plot of the function $f(x, y) = \frac{xy}{x^2+y^2}$

2. f has a limit at P .
3. $\lim_{Q \rightarrow P} f(Q) = f(P)$.

10 Partial derivatives

Given a function $f(x, y, z)$, we define the partial derivative as

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

This is the same as computing an ordinary derivative with respect to x if we imagine that y and z are fixed parameters. The derivatives with respect to y and z are defined in a similar fashion.

Example 31. For the function $f(x, y, z) = xy^2 \sin(yz)$, find all the first and second order partial derivatives.

We can compute the first order derivatives as follows:

$$\begin{aligned} \frac{\partial f}{\partial x} &= y^2 \sin(yz) \\ \frac{\partial f}{\partial y} &= x(2y) \sin(yz) + xy^2(z \cos(yz)) \\ \frac{\partial f}{\partial z} &= xy^2(y \cos(yz)) \end{aligned}$$

We can compute the second derivative as follows:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (y^2 \sin(yz)) = 0$$

We can similarly compute $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial z^2}$. We can also compute the mixed derivatives as follows:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = 2y \sin(yz) + y^2 z \cos(yz)$$

Note that the order of the derivatives does not matter and we have the important result:

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

We can check that this holds in the given example as:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 2y \sin(yz) + y^2 z \cos(yz) = \frac{\partial^2 f}{\partial x \partial y}$$

10.1 Implicit differentiation

This technique is useful when a function is given in implicit form and there is no explicit expression for it. This is illustrated by the following example.

Example 32. For the implicit function $x^3 z + y \cos z = 0$, compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

There are 2 independent variables x and y here, while z is a dependent variable which depends on x and y . Even though there is no explicit expression for z , such as $z = f(x, y)$, the solution z of the equation can be viewed as a function of x and y . Thus, to compute the partial derivative of z w.r.t. x , we consider y as fixed, and use the chain rule and the product rule to differentiate the equation:

$$\begin{aligned} \frac{\partial}{\partial x} (x^3 z + y \cos z) &= 0 \\ 3x^2 z + x^3 \frac{\partial z}{\partial x} + y(-\sin z) \frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial x} &= -\frac{3x^2 z}{x^3 - y \sin z} \end{aligned}$$

We can do the same calculation with y to get

$$\frac{\partial z}{\partial y} = -\frac{\cos z}{x^3 - y \sin z}$$

10.2 Applications

10.2.1 Partial Differential Equations

These arise in many areas, such as

1. Steady state temperature: the Laplace equation governs the variation of temperature $T(x, y)$ in space at steady state:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

2. The displacement $y(x, t)$ of a vibrating string is described by the one dimensional wave equation:

$$\frac{\partial^2 y}{\partial t^2} = a \frac{\partial^2 y}{\partial x^2}$$

where the coefficient a depends on density and tension.

3. The temperature $T(t, x, y)$ of an insulated thin plate follows the heat equation:

$$\frac{\partial T}{\partial t} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

4. A surface such that it has the least surface area for a given boundary is called a *minimal surface*. This is the case for soap bubbles for example. Then $z = f(x, y)$ must satisfy

$$(1 + (z_y)^2) z_{xx} + (1 + (z_x)^2) z_{yy} = 2z_x z_y z_{xy}$$

where we have used the notation $z_x = \frac{\partial z}{\partial x}$. A minimal surface for a helical boundary is shown in Figure 45.

5. Approximating a function in the vicinity of a point.
6. Finding the minimum or maximum of a multi-variable function.

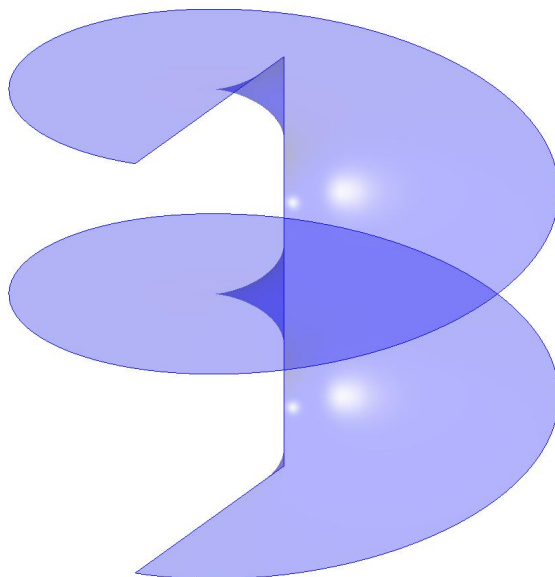


Figure 45: A helicoid, which is a minimal surface for a helical boundary

10.3 Chain rule of differentiation

The chain rule for multiple variables is similar to the chain rule for a single variable which says that if we have two functions $f(x)$ and $x(t)$, then

$$\frac{d}{dt} [f(x(t))] = \frac{df}{dx} \frac{dx}{dt}$$

For a function in 3 dimensions, we can state the chain rule as:

$$\frac{d}{dt} [f(x(t), y(t), z(t))] = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} + \frac{df}{dz} \frac{dz}{dt}$$

Example 33. *There is a melting block of ice which is in the shape of a cylinder with height $h = 40$ cm and radius $r = 15$ cm. Also, $\frac{dh}{dt} = -3$ and $\frac{dr}{dt} = -1$. Find $\frac{dV}{dt}$.*

We have the volume as $V = \pi r^2 h$ which can be differentiated as

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} \\ &= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt} \\ &= \pi 15 (2 \cdot 40 \cdot (-1) + 15 \cdot (-3)) \\ &\approx -5890 \text{ cm}^3/\text{h} \end{aligned}$$

Given a function $f(x, y, z)$ where x , y and z depend on u and v , i.e., we have functions $x(u, v)$, $y(u, v)$ and $z(u, v)$, the chain rule of differentiation gives:

$$\frac{\partial}{\partial u} [f(x(u, v), y(u, v), z(u, v))] = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$$

The derivative with respect to v can be computed in a similar fashion.

Example 34. *Given a function $f(x, y)$ and $x = r \cos \theta$, $y = r \sin \theta$, compute $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.*

Using the chain rule, we get:

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial}{\partial r} [f(x(r, \theta), y(r, \theta))] \\ &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \\ &= \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \end{aligned} \tag{3}$$

Similarly:

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= \frac{\partial}{\partial \theta} [f(x(r, \theta), y(r, \theta))] \\ &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\ &= -r \sin \theta \frac{\partial f}{\partial x} + r \cos \theta \frac{\partial f}{\partial y} \end{aligned} \tag{4}$$

To find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, we can solve for these in terms of $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ using equations (3) and (4). This is easier than re-computing these using the chain rule. We obtain:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial y} &= \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} \end{aligned}$$

10.4 Implicit differentiation

We can now return to the problem of implicit differentiation and apply the chain rule we just learned. Given an implicit function as $F(x, y) = 0$, we want to compute $\frac{dy}{dx}$. We can differentiate this directly and use the chain rule to get

$$\frac{dF}{dx} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

which simplifies to give the important result:

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}} \quad (5)$$

Example 35. *Folium of Descartes.* Given the implicit function $F(x, y) = x^3 + y^3 - 3xy = 0$, compute $\frac{dy}{dx}$. This function is plotted in Figure 46.

We have

$$F_x = 3x^2 - 3y$$

and also

$$F_y = 3y^2 - 3x$$

Thus, using equation (5), we have:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{x^2 - y}{y^2 - x}$$

From this result, we conclude that the slope is zero when $y = x^2$ and is infinite when $x = y^2$. Look at Figure 46 to verify this is correct.

Descartes was first to discuss this curve, which he discovered in an attempt to challenge Fermat's extremum-finding techniques, in 1638. Although Descartes found the correct shape of the curve in the positive quadrant, he believed that this leaf shape was repeated in each quadrant like the four petals of a flower. Descartes challenged Fermat to find the tangent line at arbitrary points. Fermat achieved success immediately, much to the chagrin of Descartes. In French the Folium of Descartes (leaf in latin) is sometimes called the *nœud de ruban*.

Reading: Thomas - 14.5, 14.6, 14.7, 14.8, 14.9

11 Partial derivatives

11.1 Linearization

The basic idea of linearization is to approximate a given function $f(x)$ by a linear function (a line) $f(x_0) + f'(x_0)(x - x_0)$ in the neighborhood of a point x_0 as shown in figure 47.

We can similarly approximate a multi-variable function $f(x, y)$ by a tangent plane in the neighborhood of a point (x_0, y_0) as (see figure 48):

$$f(x, y) = f(x_0, y_0) + (x - x_0)f_x + (y - y_0)f_y$$

If we denote the function $f(x, y)$ by z , then we get the equation of the tangent plane by rearranging the above equation as

$$z - z_0 = (x - x_0)f_x + (y - y_0)f_y$$

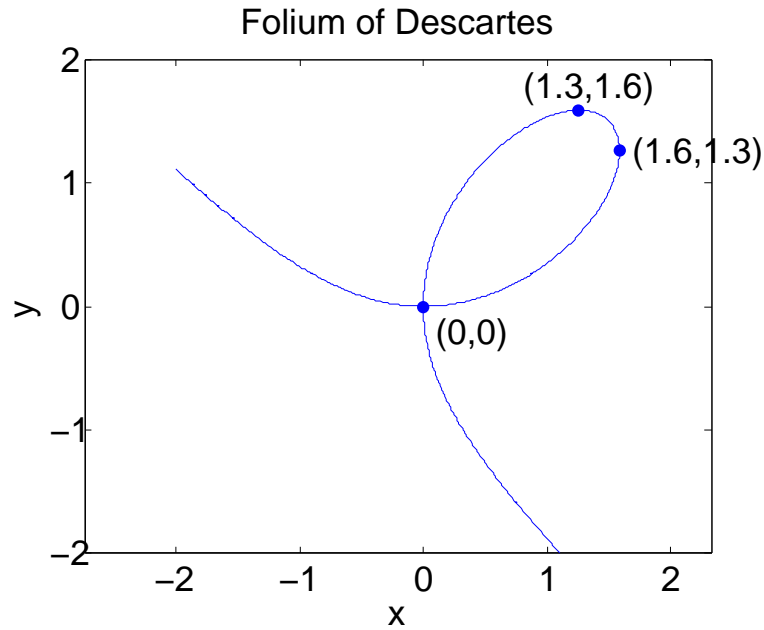


Figure 46: Descartes' folium

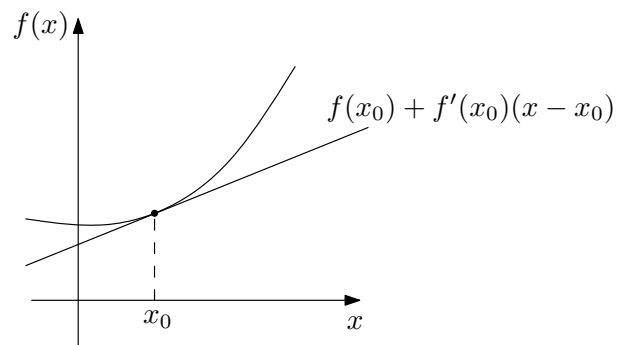


Figure 47: Linearizing a 1D function near a point

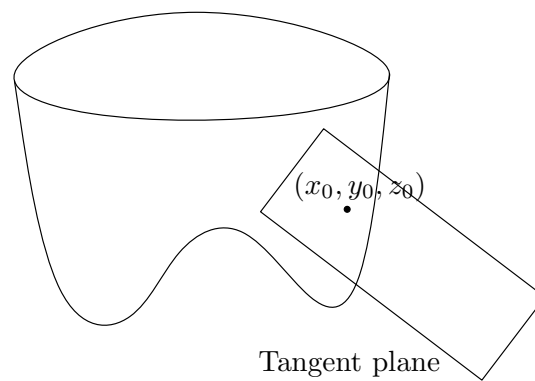


Figure 48: Linearizing a 2D function near a point

We can rewrite this to obtain an equation in the “usual” form:

$$f_x x + f_y y - z = f_x x_0 + f_y y_0 - z_0$$

We can now see that the normal to the plane is given by $\vec{n} = (f_x, f_y, -1)$.

The idea of linearization can be generalized to functions in any dimension N :

$$f(x_1, \dots, x_N) = f(x_1^0, \dots, x_N^0) + (x_1 - x_1^0)f_{x_1}(x_1^0, \dots, x_N^0) + \dots + (x_N - x_N^0)f_{x_N}(x_1^0, \dots, x_N^0)$$

Example 36. Two resistors R_1 and R_2 are connected in parallel. If $R_1 = 200 \Omega$ and $R_2 = 300 \Omega$ and if these values may be in error by as much as 2Ω , estimate the maximum error in the calculated value of the combined resistance.

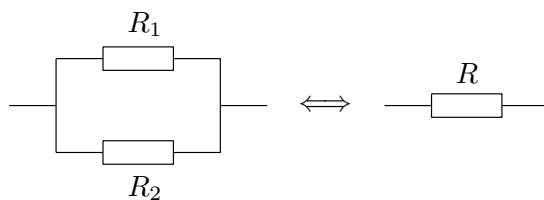


Figure 49: The resistor R is equivalent to the two resistors R_1 and R_2 connected in parallel

The combined resistance R of the parallel resistors R_1 and R_2 is given by:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

We can differentiate this w.r.t. R_1 to get

$$-\frac{1}{R^2} \frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}$$

which gives us

$$\frac{\partial R}{\partial R_1} = \left(\frac{R}{R_1}\right)^2$$

Similarly, we can differentiate w.r.t. R_2 :

$$\frac{\partial R}{\partial R_2} = \left(\frac{R}{R_2}\right)^2$$

Now, using our linearization formula for a function of two variables, we get

$$\Delta R \approx \left(\frac{R}{R_1}\right)^2 \Delta R_1 + \left(\frac{R}{R_2}\right)^2 \Delta R_2$$

Using the values $R_1 = 200 \Omega$, $R_2 = 300 \Omega$, and $\Delta R_1 = \Delta R_2 = 2 \Omega$, we calculate $R = 120 \Omega$ and use the linearization to get:

$$|\Delta R| \leq \left(\frac{120}{200}\right)^2 2 + \left(\frac{120}{300}\right)^2 2 = 1.04 \Omega$$

Example 37. Compute the partial derivative of the volume of a cylinder with respect to its height and radius. Perform a linearization around $r = 1$, $h = 5$.

We know that the volume of a cylinder is given by $V = \pi r^2 h$. This can be linearized using the partial derivatives w.r.t. r and h :

$$\Delta V \approx \pi [2rh\Delta r + r^2\Delta h]$$

For $r = 1$ and $h = 5$, we get

$$\Delta V \approx \pi [10\Delta r + \Delta h]$$

If we reduce the radius by ε , i.e., $\Delta r = -\varepsilon$ and increase the height by 10ε , i.e., $\Delta h = 10\varepsilon$, the change in volume is:

$$\Delta V = \pi [-10\varepsilon + 10\varepsilon] = 0$$

The volume is roughly unchanged! This shows that it is 10 times more sensitive to the radius than the height!

11.2 Directional derivative and gradient

11.2.1 Directional derivative

Given a function $f(x, y)$, if we compute the derivative along \vec{i} (x -axis) keeping y constant then we get $\frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the derivative in the direction along \vec{j} (y -axis) is given by $\frac{\partial f}{\partial y}$. We can also think of the derivative along an arbitrary unit vector $\vec{u} = (u_1, u_2)$ (see figure 50) as being given by

$$\mathcal{D}_{\vec{u}}f = \lim_{h \rightarrow 0} \frac{f(x+u_1h, y+u_2h) - f(x, y)}{h}$$

where $\mathcal{D}_{\vec{u}}f$ denotes the directional derivative of the function f along the unit vector \vec{u} .

To obtain a more explicit expression, define the function $g(h) = f(x_0 + u_1h, y_0 + u_2h)$ which can be differentiated to get

$$\frac{dg}{dh} = \frac{\partial f}{\partial x} \frac{d}{dh}(x_0 + u_1h) + \frac{\partial f}{\partial y} \frac{d}{dh}(y_0 + u_2h) = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

Thus, the directional derivative can be computed as

$$\mathcal{D}_{\vec{u}}f = \frac{dg}{dh}(0) = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

11.2.2 Gradient

The vector $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is called the gradient of f and is denoted by $\vec{\nabla}f$. The gradient is also sometimes written as ∇f (the notation that we will use). You should note that it is a **vector**.

The definition of the directional derivative gives us the following relationship with the gradient:

$$\mathcal{D}_{\vec{u}}f = \nabla f \cdot \vec{u}$$

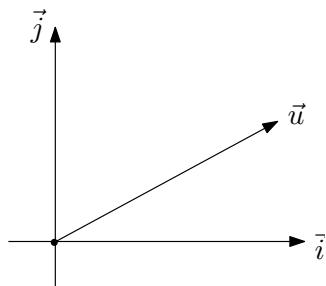


Figure 50: Defining the directional derivative along a unit vector \vec{u}

11.2.3 Gradient properties

An important property of the gradient is that

$$-|\nabla f| \leq \mathcal{D}_{\vec{u}}f \leq |\nabla f|$$

In addition, $\mathcal{D}_{\vec{u}}f$ is maximum when \vec{u} points in the direction of ∇f and is minimum when \vec{u} points in the opposite direction. Note that:

1. $\mathcal{D}_{\vec{u}}f = 0$ if \vec{u} is perpendicular to ∇f .
2. ∇f is perpendicular to the level curves.
3. ∇f points in the direction of maximum increase of f .

The function $z = f(x, y) = y^2 - x^2$ is plotted in Figure 51 and its level curves and gradient are plotted in Figure 52.

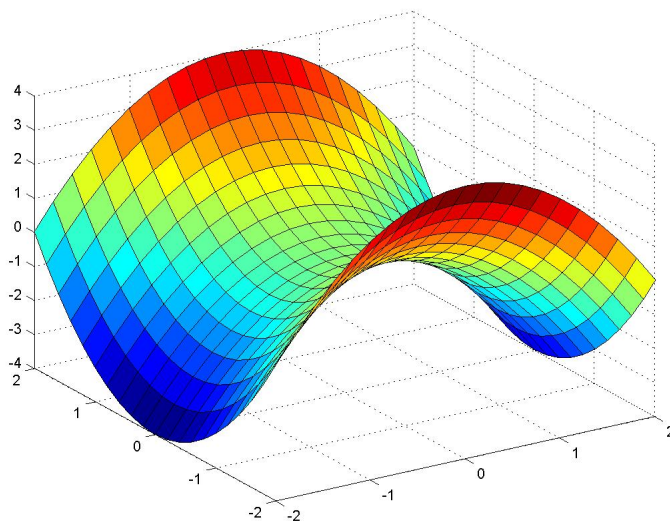


Figure 51: A MATLAB plot of the surface $f(x, y) = y^2 - x^2$

Example 38. For the function $z = f(x, y) = 10 - x^2 - y^2$, determine:

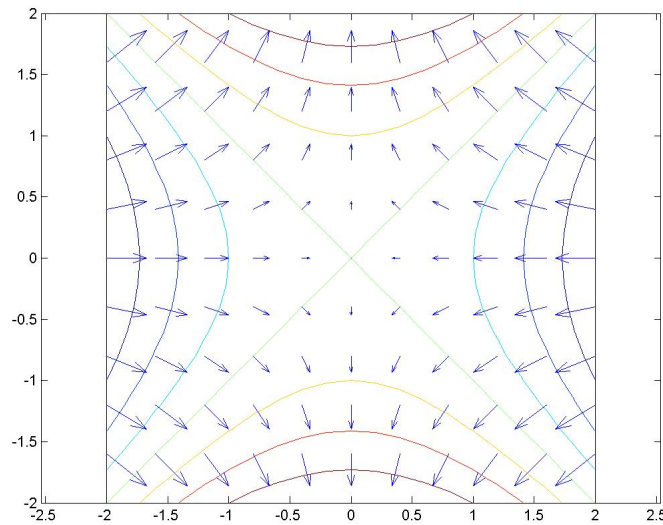


Figure 52: Plotting the gradient vectors and the level curves of the surface $f(x, y) = y^2 - x^2$. Note that the vectors are normal to the level curves.

1. The gradient of z at $(1,1)$
2. The directional derivative in the direction of $\vec{u} = \frac{3\vec{i}+4\vec{j}}{5}$ at $(1,1)$
3. The directions of maximum increase, maximum decrease, and no change in z
4. A particle moves by a distance of 0.1 units in the direction of the gradient. Determine the approximate change in z .
5. A vector normal to the surface $z = 10 - x^2 - y^2$ at $(1,1)$

1. The gradient of z at $(1,1)$:
We can compute the gradient as $\nabla f = (-2x, -2y)$. At $(1,1)$ this computes to $\nabla f = (-2, -2)$.
2. The directional derivative in the direction of $\vec{u} = \frac{3\vec{i}+4\vec{j}}{5}$ at $(1,1)$:
We have

$$\mathcal{D}_{\vec{u}}f = \nabla f \cdot \vec{u} = (-2, -2) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = -\frac{14}{5}$$
3. The directions of maximum increase, maximum decrease, and no change in z :
The direction of maximum increase is given by $\frac{1}{\sqrt{2}}(-1, -1)$ and the direction of maximum decrease is given by $\frac{1}{\sqrt{2}}(1, 1)$. Directions in which the function does not change are $\frac{1}{\sqrt{2}}(-1, 1)$ and $\frac{1}{\sqrt{2}}(1, -1)$.
4. A particle moves by a distance of 0.1 units in the direction of the gradient. Determine the approximate change in z :

We can linearize z to obtain:

$$\begin{aligned}
 \Delta z &= \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \\
 &= \nabla f \cdot (\Delta x, \Delta y) \\
 &= |\nabla f| |(\Delta x, \Delta y)| \cos \theta \\
 &= |\nabla f| d \\
 &= \sqrt{8} 0.1 \\
 &\approx 0.3
 \end{aligned}$$

5. A vector normal to the surface $z = 10 - x^2 - y^2$ at $(1,1)$:

The surface is defined by the $z = f(x, y)$, where the function $f(x, y)$ is $10 - x^2 - y^2$. Now, we can compute the tangent plane to the surface at a point (x_0, y_0) using the equation of the tangent plane derived at the beginning of this lecture. We also computed a vector normal to the tangent plane at the point (x_0, y_0) as $(f_x(x_0, y_0), f_y(x_0, y_0), -1)$. Therefore, the unit normal vector to the surface at the point $(1, 1)$ is $-\frac{1}{3}(2, 2, 1)$.

11.2.4 Application

The gradient is very useful for numerically finding the maximum and the minimum of functions. For example, to find the maximum of a function, you can start from a point (x_0, y_0) and keep taking a small step in the direction of the gradient until a maximum is found. This leads to the important problem of optimization which is the goal of many engineering projects.

12 Optimization

Optimization is concerned with finding the extremum (maximum or minimum) of functions.

12.1 Definition of maximum and minimum

Local minimum: A function $f(x)$ has a local minimum at point x_0 if the value of $f(x)$ at all points x in the immediate neighborhood of x_0 is greater than $f(x_0)$.

Global minimum: A function $f(x)$ has a global minimum at point x_0 if the value of $f(x)$ at any point in the entire domain of $f(x)$ is greater than $f(x_0)$.

Local maximum: A function $f(x)$ has a local maximum at point x_0 if the value of $f(x)$ at all points x in the immediate neighborhood of x_0 is less than $f(x_0)$.

Global maximum: A function $f(x)$ has a global maximum at point x_0 if the value of $f(x)$ at any point in the entire domain of $f(x)$ is less than $f(x_0)$.

Figure 53 illustrates these different definitions.

We can formulate three conditions which must be met (necessary conditions) for a 2D function $f(x, y)$ to have an extremum at the point (a, b) :

1. The point (a, b) lies inside the domain and

$$f_x = f_y = 0$$

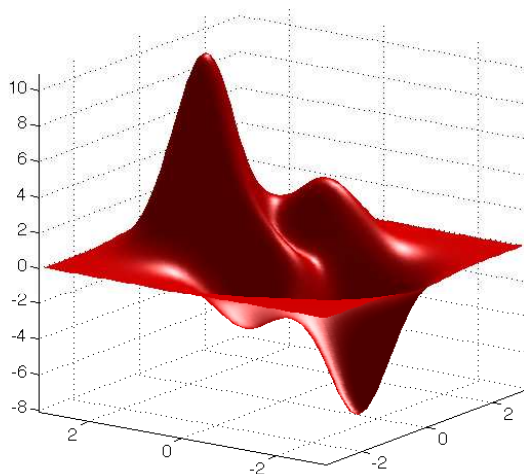


Figure 53: This function has local and global maxima and minima

2. f_x or f_y is not defined.
3. The point (a, b) is on the boundary.

Example 39. Find the extremum of the function $f(x, y) = x^2 + y^2$.

Given the function, we can compute $f_x = 2x$ and $f_y = 2y$. We can check that the function is smooth and the domain includes the entire real plane. Hence, the conditions $f_x = 0$ and $f_y = 0$ give us the point $(x, y) = (0, 0)$ as the global minimum. The global maximum is at infinity.

Example 40. Find the extremum of the function $f(x, y) = \sqrt{x^2 + y^2}$ in the region defined by $0 \leq x^2 + y^2 \leq 1$.

We can check that the function is smooth (except at the point $(0, 0)$) and the domain is given to us. And we can compute the partial derivatives as

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

At the point $(0, 0)$ the derivatives are not defined and the function has a global minimum at that point. The function attains its global maximum at all points that lie on the unit circle $x^2 + y^2 = 1$. The function is plotted in Figure 54. Note that the level curves of the function are circles and thus the boundary of the given domain exactly matches one of the level curves: $x^2 + y^2 = 1$.

Example 41. Find the extremum of the function $f(x, y) = x^2 - y^2$.

Note that there are no global extremum for this function as we can put $y = 0$ and take x as large as we want to have $f = \infty$. Similarly, we can also have $f = -\infty$ at $(0, \infty)$. Now, we will look for local extremum and compute $f_x = 2x$ and $f_y = -2y$. Note that putting $f_x = f_y = 0$ gives us the point $(0, 0)$. But at that point the function has neither a maximum nor a minimum: such a point is called a saddle point. This can be visualized in Figure 55 where two level curves intersect at the point $(0, 0)$ and the function increases in the region on the left and right of these curves, while it decreases in the region on the top and bottom, indicating that the function has a saddle point at $(0, 0)$.

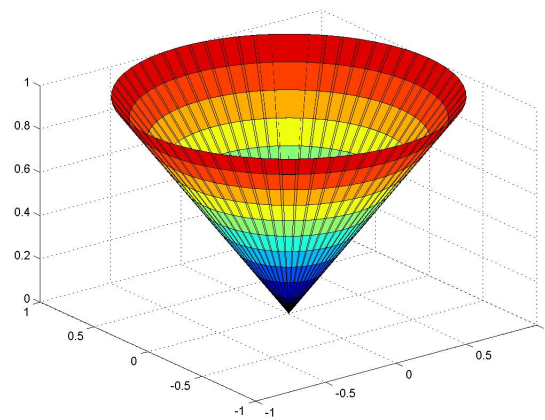


Figure 54: A MATLAB plot of the surface $f(x, y) = \sqrt{x^2 + y^2}$

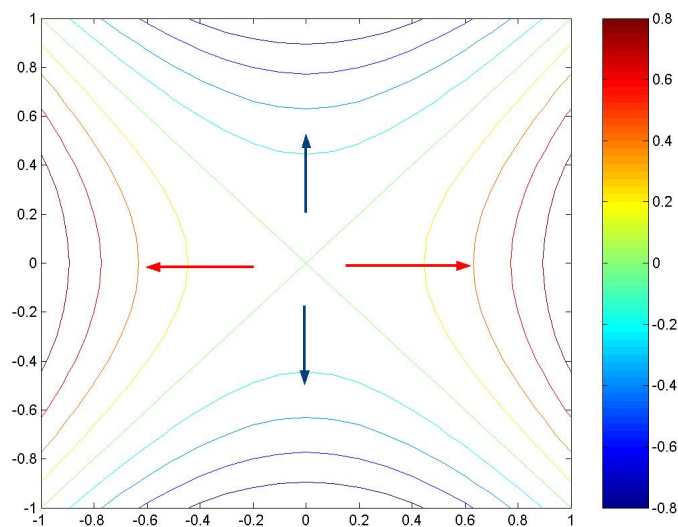


Figure 55: Level curves of the surface $f(x, y) = x^2 - y^2$. The blue arrows indicate the direction in which the function decreases and the red arrows indicate the direction in which the function increases.

12.2 Second derivative test

Assume we have a function $f(x, y)$ with $f_x = f_y = 0$ at the point (a, b) (called a singular point). We denote: $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$, and define $\Delta = AC - B^2$. This test enumerates all the possibilities at (a, b) :

1. $\Delta > 0$ and $A > 0$: the function has a local minimum at (a, b) .
2. $\Delta > 0$ and $A < 0$: the function has a local maximum at (a, b) .
3. $\Delta < 0$: the function has a saddle point at (a, b) .
4. $\Delta = 0$: the test provides no information. Higher order derivatives are needed (think about x^4 at $x = 0$).

Example 42. Show that $f(x, y) = x^2 - y^2$ has a saddle point at $(0, 0)$.

We have $f_x = 2x$, $f_y = -2y$ and $f_x = f_y = 0$ at the point $(0, 0)$. Now, we can compute $A = f_{xx} = 2$, $B = f_{xy} = 0$ and $C = f_{yy} = -2$. Thus, $\Delta = 2(-2) - 0^2 = -4$: we have a saddle point at $(0, 0)$!

Example 43. Find the least amount of plywood needed to construct a closed rectangular box of a given volume V (see Figure 56).

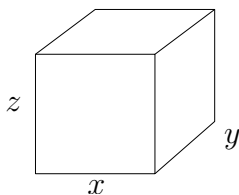


Figure 56: Calculating the plywood needed to construct a box

The surface area of the box is given by $S = 2xz + 2xy + 2yz$ and the volume is given by $V = xyz$. Thus, we have $z = \frac{V}{yz}$ and can put this in the expression for the surface area to get

$$\begin{aligned} S &= 2x \left(\frac{V}{xy} \right) + 2xy + 2y \left(\frac{V}{xy} \right) \\ &= 2 \left(\frac{V}{y} + xy + \frac{V}{x} \right) \end{aligned}$$

We can compute the partial derivatives and set them equal to zero:

$$\begin{aligned} \frac{\partial S}{\partial x} &= -\frac{2V}{x^2} + 2y = 0 \\ \frac{\partial S}{\partial y} &= -\frac{2V}{y^2} + 2x = 0 \end{aligned}$$

These equations give us the following relationships:

$$\begin{aligned} V &= x^2y \\ V &= xy^2 \end{aligned}$$

and consequently, we have $x = y$. We finally get that $x = y = z = V^{\frac{1}{3}}$ is the only critical point. Now, to determine if this critical point is a minimum, we need to examine the second derivatives according to the test described earlier. Computing the derivatives, we have $A = S_{xx} = \frac{4V}{x^3} = 4$ and similarly, $B = 2$ and $C = 4$. Thus, we can compute

$$\Delta = AC - B^2 = 16 - 4 = 12$$

Since $\Delta > 0$ and $A > 0$, we can see that $(V^{\frac{1}{3}}, V^{\frac{1}{3}}, V^{\frac{1}{3}})$ is the local minimum and hence the least amount of plywood is needed when all three dimensions of the box are equal.

Example 44. Determine an equation of the line that best fits n data points $(x_1, y_1), \dots, (x_n, y_n)$ in the least squares sense.

Suppose that we have a line with slope a and intercept b . Then, the error (see Figure 57) in the i^{th} measurement is given by

$$E_i = ax_i + b - y_i$$

where (x_i, y_i) is the i^{th} measurement.

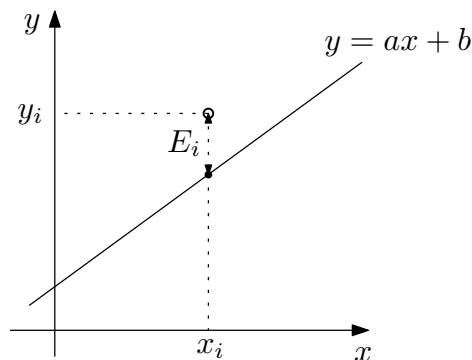


Figure 57: Error in a measurement

Since we want the “best” fit of the line, our objective can be stated as minimizing the sum of all the errors. For the least square approximation, we use the following function to be minimized:

$$E = \sum_{i=1}^n E_i^2 = \sum_{i=1}^n (ax_i + b - y_i)^2$$

Thus, our objective is to solve the following optimization problem:

$$\text{Minimize over } a \text{ and } b: \quad E = \sum_{i=1}^n (ax_i + b - y_i)^2$$

We find the minimum by solving for $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$:

$$\begin{aligned} \frac{\partial E}{\partial a} &= 2 \sum_{i=1}^n x_i (ax_i + b - y_i) = 0 \\ \Rightarrow \quad a \sum_{i=1}^n (x_i)^2 + b \sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i &= 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned}\frac{\partial E}{\partial b} &= -2 \sum_{i=1}^n (ax_i + b - y_i) = 0 \\ \Rightarrow a \sum_{i=1}^n x_i + nb - \sum_{i=1}^n y_i &= 0\end{aligned}\tag{7}$$

Solving Equations 6 and 7 for a and b , we get:

$$\begin{aligned}a &= \frac{(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i) - n \sum_{i=1}^n x_i y_i}{(\sum_{i=1}^n x_i)^2 - n \sum_{i=1}^n x_i^2} \\ b &= \frac{1}{n} \left(y_i - a \sum_{i=1}^n x_i \right)\end{aligned}$$

An example of solving for such a line is shown in Figure 58.

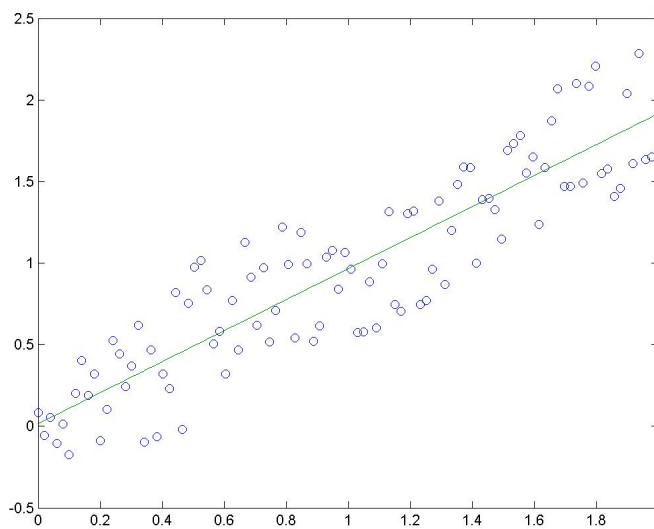


Figure 58: Computing the least squares line for given points (shown as circles)

Example 45. The temperature over a semicircular disk of radius 1 is given by: $T(x, y) = xy + x^2$. Find the absolute maxima and minima (if any) over the specified region.

We first look for the critical points of the function T inside the semicircular disk using the equations:

$$\begin{aligned}\frac{\partial T}{\partial x} &= y + 2x = 0 \\ \frac{\partial T}{\partial y} &= x = 0\end{aligned}$$

The solution to these equations is given by $x = y = 0$.

Now, we look for extremum points on the boundary. For the diameter of the disk, i.e., $-1 \leq x \leq 1$ and $y = 0$, we have $T = x^2$. By inspection, we can see that the minimum of this function is $T = 0$ at $x = 0$ and the maximum of the function is $T = 1$ and $x = \pm 1$.

For the circular part, we can parameterize the coordinates as $x = \cos \theta$ and $y = \sin \theta$. Thus we have

$$\begin{aligned} T(\theta) &= \cos \theta \sin \theta + \cos^2 \theta \\ &= \frac{1}{2} (1 + \cos(2\theta) + \sin(2\theta)) \end{aligned}$$

We can now find the critical points of this 1D function by solving $\frac{dT}{d\theta} = 0$ which gives

$$\begin{aligned} -2 \sin(2\theta) + 2 \cos(2\theta) &= 0 \\ \tan(2\theta) &= \pm 1 \end{aligned}$$

The solution is given by:

$$2\theta = \frac{\pi}{4}, \frac{5\pi}{4} \Rightarrow \theta = \frac{\pi}{8}, \frac{5\pi}{8}$$

We can compute the value of the function at these points:

$$\begin{aligned} T\left(\frac{\pi}{8}\right) &= \frac{1 + \sqrt{2}}{2} \approx 1.2 \\ T\left(\frac{5\pi}{8}\right) &= \frac{1 - \sqrt{2}}{2} \approx 0.2 \end{aligned}$$

Thus the absolute maximum value of the function is $\frac{1+\sqrt{2}}{2}$ which occurs at the point $(\cos \frac{\pi}{8}, \sin \frac{\pi}{8})$ and the absolute minimum value of the function is $\frac{1-\sqrt{2}}{2}$ which occurs at the point $(\cos \frac{5\pi}{8}, \sin \frac{5\pi}{8})$. The minimum and the maximum are shown in Figure 59.

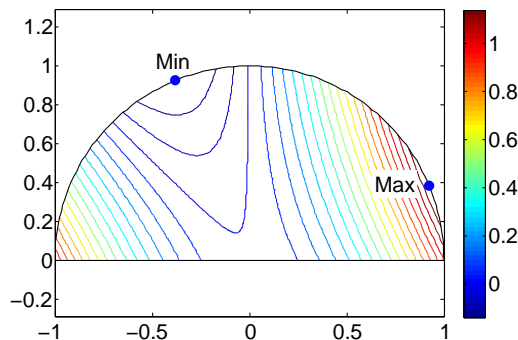


Figure 59: The minimum and the maximum temperature over the semi-circular region

13 Optimization with constraints

Let us say we want to maximize a function $f(x, y)$ with the constraint that $g(x, y) = 0$. Further, assume that the maximum is achieved at the point (x_0, y_0) ; take a parametrization $(x(t), y(t))$ such that $x(0) = x_0$ and $y(0) = y_0$. At $t = 0$, we must have

$$\frac{d}{dt} (f(x(t), y(t))) = 0$$

which gives us

$$\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0$$

Hence we have $\nabla f \cdot \vec{v} = 0$, which implies that ∇f is perpendicular to \vec{v} . Note that ∇g is also perpendicular to \vec{v} and therefore, ∇f is parallel to ∇g . Thus, we must have a scalar λ such that

$$\nabla f = \lambda \nabla g$$

Hence, we have 3 unknowns, x , y and λ , along with 3 equations:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ g(x, y) &= 0\end{aligned}$$

Example 46. Find the point on the hyperbola $xy = 1$ which is closest to the origin.

We want to minimize the distance function $\sqrt{x^2 + y^2}$, which is the same as minimizing $f(x, y) = x^2 + y^2$ subject to the constraint that $xy = 1$. We can write our 3 equations as:

$$\begin{aligned}2x &= \lambda y \\ 2y &= \lambda x \\ xy &= 1\end{aligned}$$

We can solve this system of equations to get $x = y = 1$ and $\lambda = 2$. The optimum points are shown in Figure 60. Note that the hyperbola is tangent to a circle around the origin at the optimum point.

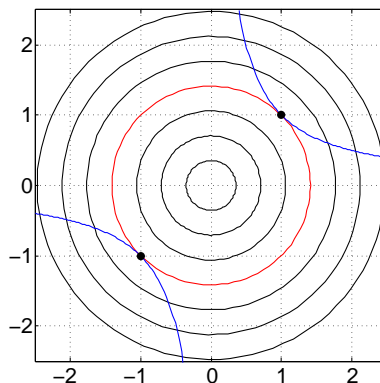


Figure 60: The closest point to the origin on the hyperbola $xy = 1$

Example 47. Suppose we want to cut a rectangular beam from a given elliptical log (with the length of the semi-major axis 2 and the length of the semi-minor axis 1). We want to maximize the area of the rectangular beam in order to extract the maximum wood from the log. Find the dimensions of the maximum area rectangular beam that can be cut from this log.

We want to minimize the area function $f(x, y) = 4xy$ subject to the constraint that $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$. We can write our 3 equations as:

$$\begin{aligned}4y &= \lambda \frac{2x}{4} \\ 4x &= \lambda 2y \\ \left(\frac{x}{2}\right)^2 + y^2 &= 1\end{aligned}$$

We can solve this system of equations to get $x = \sqrt{2}$, $y = \frac{1}{\sqrt{2}}$ and $\lambda = 4$. The situation is depicted in Figure 61. Note that the area function is a hyperbola and it is tangent to the ellipse at the optimal points.

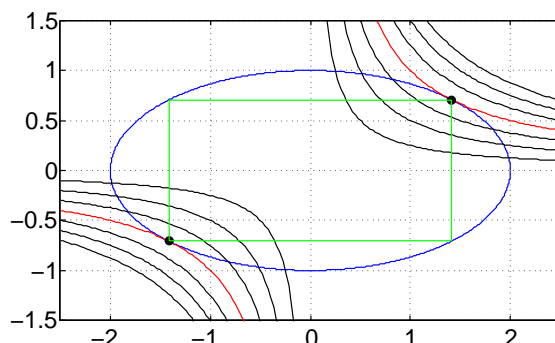


Figure 61: The maximum area rectangle that can fit in an ellipse

Example 48. *This is a problem in economics. The gross output Q can often be approximated using: $Q = AK^aL^{1-a}$ where K is the capital investment and L is the amount of labor. This problem is subject to the constraint that the total cost of operation is fixed: $M = pK + wL$. How would you choose K and L in order to optimize Q and therefore your profit? Use a Lagrange multiplier to find the solution.*

Here, $g(K, L) = pK + wL - M$. Our unknowns are K , L and λ and we can write our equations as:

$$\begin{aligned}\frac{\partial Q}{\partial K} &= \lambda \frac{\partial g}{\partial K} \\ \frac{\partial Q}{\partial L} &= \lambda \frac{\partial g}{\partial L} \\ pK + wL &= M\end{aligned}$$

The three equations simplify to

$$\frac{\partial Q}{\partial K} = \lambda p \tag{8}$$

$$\frac{\partial Q}{\partial L} = \lambda w \tag{9}$$

$$pK + wL = M \tag{10}$$

This gives us $\lambda = \frac{1}{p} \frac{\partial Q}{\partial K} = \frac{1}{w} \frac{\partial Q}{\partial L}$. This is saying that at the maximum, the marginal change in output per dollar's worth of capital equals the marginal change per dollar's worth of extra labor. In other words, exchanging labor for capital does not change the output. Now, inserting Equations (8) and (9) in Equation (10), we get:

$$\frac{1}{\lambda} \frac{\partial Q}{\partial K} K + \frac{1}{\lambda} \frac{\partial Q}{\partial L} L = M$$

Using the definition of Q , we get

$$\frac{1}{\lambda} aQ + \frac{1}{\lambda} (1-a)Q = M$$

which gives us $\lambda = \frac{Q}{M}$. Hence, from Equation (8), we get

$$\frac{\partial Q}{\partial K} = \frac{Q}{M}p$$

which implies that $\frac{\partial}{\partial K} \ln Q = \frac{p}{M}$. From the definition of $Q = AK^aL^{1-a}$, this implies that:

$$\frac{a}{K} = \frac{p}{M}$$

Similarly, from $\frac{\partial Q}{\partial L} = \frac{Q}{M}w$, we get:

$$\frac{1-a}{L} = \frac{w}{M}$$

As a conclusion, at the maximum, we have

$$K = \frac{aM}{p}, \quad L = \frac{(1-a)M}{w},$$

and $Q = AM \left(\frac{a}{p}\right)^a \left(\frac{1-a}{w}\right)^{1-a}$.

Reading: Thomas - 15.1, 15.2

14 Integration in multiple dimensions

14.1 Double integrals

Consider a function $f(x, y)$ of two variables. Given that the function is continuous, we can define the iterated integral as

$$\int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

If we define $g(x) = \int_c^d f(x, y) dy$, then we get the iterated integral as

$$\int_a^b g(x) dx$$

Note that the order of integration does not change the value of the integral and we have

$$\int_a^b \left[\int_c^d f(x, y) dy \right] dx = \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$

We can also define the integral $\iint_R f(x, y) dx dy$ over the rectangle R (see Figure 62(a)). It is defined as follows. Let's divide the region R into smaller regions as shown in Figure 62(b). Then, we get

$$\begin{aligned} x_1 &= a \\ x_2 &= a + h \\ x_i &= a + (i - 1)h \end{aligned}$$

and

$$\begin{aligned} y_1 &= c \\ y_2 &= c + h \\ y_i &= c + (i - 1)h \end{aligned}$$

The integral is defined as the following limit:

$$\iint_R f(x, y) dx dy = \lim_{h \rightarrow 0} \sum_{i=1}^m \sum_{j=1}^n h^2 f(x_i, y_j)$$

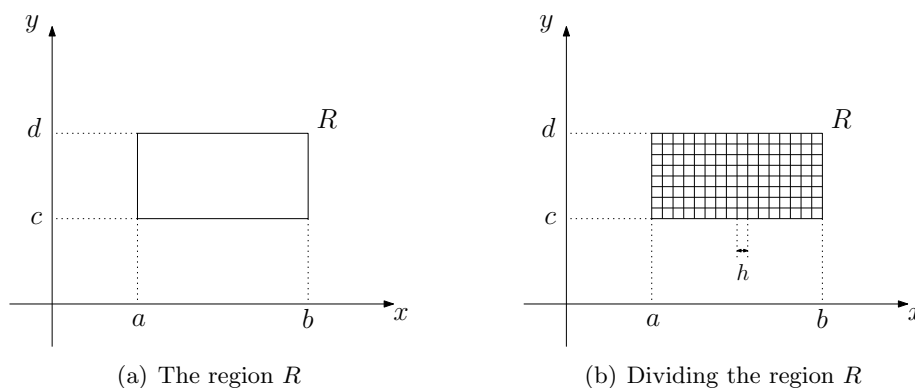


Figure 62: Integrating over a rectangular region

14.1.1 Fubini's Theorem

For a continuous function $f(x, y)$, we have

$$\begin{aligned} \iint_R f(x, y) dx dy &= \int_a^b \left[\int_c^d f(x, y) dy \right] dx \\ &= \int_c^d \left[\int_a^b f(x, y) dx \right] dy \end{aligned}$$

14.1.2 Properties

We have the following properties for the double integral:

$$\begin{aligned} \iint_R c \cdot f(x, y) dx dy &= c \cdot \iint_R f(x, y) dx dy \\ \iint_R (f(x, y) + g(x, y)) dx dy &= \iint_R f(x, y) dx dy + \iint_R g(x, y) dx dy \\ \iint_R f(x, y) dx dy &\geq 0 \text{ if } f \geq 0 \text{ everywhere in } R \\ \iint_R f(x, y) dx dy &= \iint_{R_1} f(x, y) dx dy + \iint_{R_2} f(x, y) dx dy \quad (\text{see Figure 63}) \end{aligned}$$

14.1.3 Geometric interpretation

The integral $\iint_R f(x, y) dx dy$ gives the volume under the surface $z = f(x, y)$ and the x - y plane, as shown in Figure 64.

Example 49. Given the surface $z(x, y) = x^2 + y$, find the volume under the surface for the region $0 \leq x \leq 1$ and $1 \leq y \leq 2$.

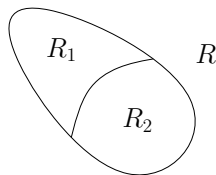


Figure 63: Integration over an arbitrary region with subdivision

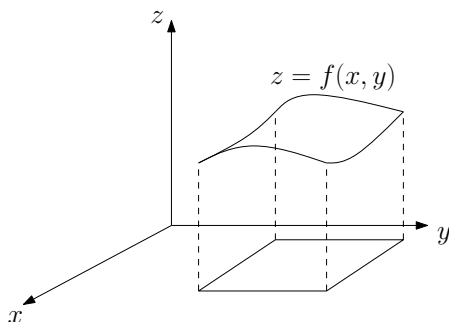


Figure 64: Integration gives the volume under a surface

We can compute the double integral as

$$\begin{aligned}
 \iint_R z(x, y) dx dy &= \int_0^1 \left[\int_1^2 (x^2 + y) dy \right] dx \\
 &= \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_1^2 dx \\
 &= \int_0^1 \left[2x^2 + \frac{4}{2} - x^2 - \frac{1}{2} \right] dx \\
 &= \int_0^1 \left[x^2 + \frac{3}{2} \right] dx \\
 &= \frac{x^3}{3} + \frac{3x}{2} \Big|_0^1 \\
 &= \frac{1}{3} + \frac{3}{2} \\
 &= \frac{11}{6}
 \end{aligned}$$

14.1.4 Extensions to integrals over arbitrary regions

We can extend our definitions of the integral to vertically simple regions as shown in Figure 65(a) and horizontally simple region as shown in Figure 65(b).

14.1.5 Extension to Fubini's theorem

If $f(x, y)$ is vertically simple, then

$$\iint_R f(x, y) dx dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y) dy \right] dx$$

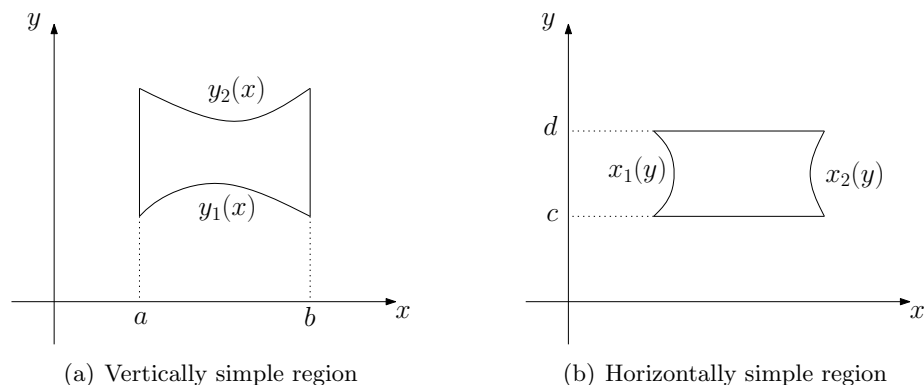


Figure 65: Defining regions

If $f(x, y)$ is horizontally simple, then

$$\iint_R f(x, y) dx dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y) dx \right] dy$$

Advice: Always plot the region of integration to decide between these cases.

Example 50. Find the area of the triangle formed by $(0,0)$, $(1,0)$ and $(1,1)$ using a double integral (see Figure 66).

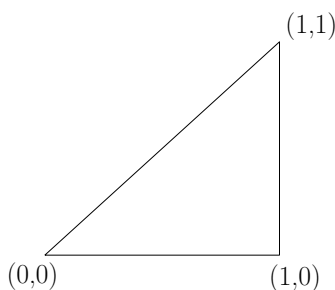


Figure 66: The given triangle

We can see that the triangle is both vertically simple and horizontally simple. Let's compute the area A using the formula for a vertically simple region as:

$$A = \int_0^1 \left[\int_0^x dy \right] dx = \int_0^1 x dx = \frac{1}{2}$$

Example 51. Find the volume of the prism whose base is the triangle in the x - y plane bounded by the x -axis and the lines $y = x$ and $x = 1$, and whose top lies in the plane $z = 3 - x - y$.

We can plot this triangle as shown in Figure 66. We want to compute the integral

$$\begin{aligned} \int_0^1 \left[\int_0^x (3-x-y) dy \right] dx &= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_0^x dx \\ &= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx \\ &= \frac{3}{2} - \frac{1}{3} - \frac{1}{6} \\ &= \frac{9-2-1}{6} \\ &= 1 \end{aligned}$$

You should check that reversing the order of the integral results in the same answer. Note that the reverse integral would be

$$\int_0^1 \left[\int_y^1 (3-x-y) dx \right] dy$$

Example 52. Calculate $\iint_R \frac{\sin x}{x} dA$ over the triangle in the xy -plane bounded by the x -axis and the lines $y = x$ and $x = 1$.

We want to compute the integral

$$\begin{aligned} \int_0^1 \left[\int_0^x \frac{\sin x}{x} dy \right] dx &= \int_0^1 \left[\frac{\sin x}{x} \right]_0^x dx \\ &= \int_0^1 \left(\frac{\sin x}{x} \right) dx \\ &= 1 - \cos(1) \end{aligned}$$

Note that computing the reverse integral in this case turns out to be impossible:

$$\int_0^1 \left[\int_y^1 \frac{\sin x}{x} dx \right] dy$$

14.2 Applications of double integrals

14.2.1 Computing the mass

Let's say we have the density $\delta(x, y)$ function over an object and want to compute its mass (see Figure 67). Recall that density gives us mass per unit area, and hence the mass can be computed as

$$M = \iint_R \delta(x, y) dx dy$$

14.2.2 Centroid / Center of mass

Recall that the location of the center of mass for a finite number of objects in 1D is defined as

$$\bar{x} = \frac{\sum_{i=1}^n x_i m_i}{\sum_{i=1}^n m_i}$$

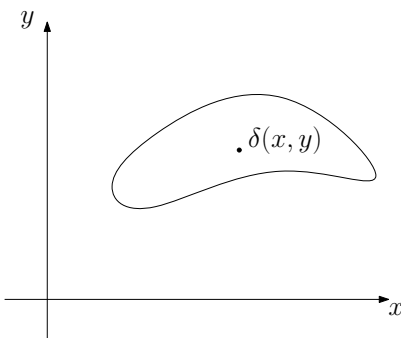


Figure 67: Given density of a lamina

For a continuous object, the location of the center of mass in 1D is given by

$$\bar{x} = \frac{\int x\delta(x)dx}{\int \delta(x)dx}$$

In 2D, this expression is given by

$$\bar{x} = \frac{\iint_R x\delta(x, y)dxdy}{\iint_R \delta(x, y)dxdy}$$

$$\bar{y} = \frac{\iint_R y\delta(x, y)dxdy}{\iint_R \delta(x, y)dxdy}$$

Example 53. A thin plate covers the triangular region bounded by the x -axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The density is given by $\delta(x, y) = 6x + 6y + 6$. Find the mass and the center of mass of the plate (see Figure 68).

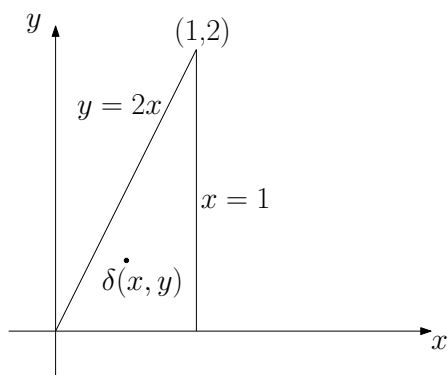


Figure 68: The thin plate

We can compute the mass as follows

$$\begin{aligned}
 \int_0^1 \left[\int_0^{2x} (6x + 6y + 6) dy \right] dx &= \int_0^1 \left[\int_0^{2x} (6x + 6y + 6) dy \right] dx \\
 &= \int_0^1 [6xy + 3y^2 + 6y]_0^{2x} dx \\
 &= \int_0^1 (12x^2 + 12x^2 + 12x) dx \\
 &= [8x^3 + 6x^2]_0^1 \\
 &= 14
 \end{aligned}$$

To compute the center of mass, also we need to compute

$$\begin{aligned}
 \iint_R x\delta(x, y) dx dy &= \int_0^1 \left[\int_0^{2x} x(6x + 6y + 6) dy \right] dx \\
 &= \int_0^1 \left[\int_0^{2x} (6x^2 + 6xy + 6x) dy \right] dx \\
 &= \int_0^1 [6x^2y + 3xy^2 + 6xy]_0^{2x} dx \\
 &= \int_0^1 (12x^3 + 12x^3 + 12x^2) dx \\
 &= [6x^3 + 4x^2]_0^1 \\
 &= 10
 \end{aligned}$$

Thus, using the definitions above, we get $\bar{x} = \frac{10}{14}$. Similarly, we can compute $\iint_R y\delta(x, y) dx dy$ and find $\bar{y} = \frac{11}{14}$.

14.3 Applications of double integrals

14.3.1 Moment of Inertia

Let's say that a body having mass m is rotating around the origin with angular velocity ω . For a small part of the body (see Figure 69) having mass Δm , located at the distance r from the origin, the speed is given by $v = \omega r$, and the kinetic energy is given by $K = \frac{1}{2}(\Delta m)v^2 = \frac{1}{2}(\Delta m)\omega^2 r^2$. Thus, for the body, we can define the total kinetic energy as:

$$K = \iint_R \frac{1}{2} \delta(x, y) \omega^2 r^2 dx dy$$

where r is the distance between (x, y) and the center of rotation, i.e., $r^2 = x^2 + y^2$. Since the angular velocity ω is same for the entire body, we denote the **polar moment of inertia** I as:

$$I = \iint_R \delta(x, y) r^2 dx dy$$

to get

$$K = \frac{1}{2} I \omega^2$$

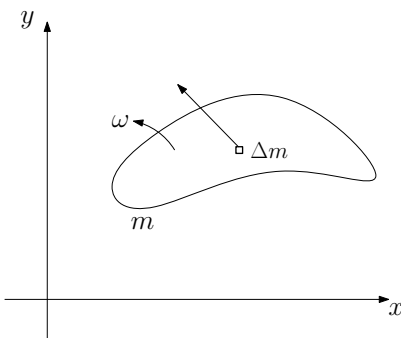


Figure 69: A body rotating around the origin

It is also possible to define the **moment of inertia about the x -axis** as

$$I_x = \iint_R \delta(x, y) y^2 dx dy$$

and about the **y -axis** as

$$I_y = \iint_R \delta(x, y) x^2 dx dy$$

Example 54. A thin plate covers the triangular region bounded by the x -axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The density is given by $\delta(x, y) = 6x + 6y + 6$. Find the moment of inertia about the x -axis I_x for the plate (see Figure 70).

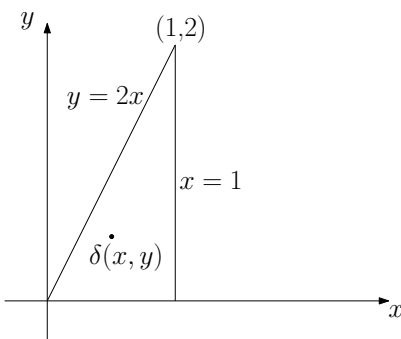


Figure 70: The thin plate

The moment of inertia can be computed as

$$\begin{aligned} I_x &= \iint_R \delta(x, y) y^2 dx dy \\ &= \iint_R (6x + 6y + 6) y^2 dx dy \\ &= \iint_R (6xy^2 + 6y^3 + 6y^2) dx dy \end{aligned}$$

This can be evaluated to get the final answer: $I_x = 12$.

14.3.2 Application of moment of inertia: I-beams

The beams used for construction have an I-shaped cross-section and we want to analyze the properties of such beams. This shape is chosen as it is particularly strong under certain types of stress. Here, we analyze the I-beam shown in Figure 71.

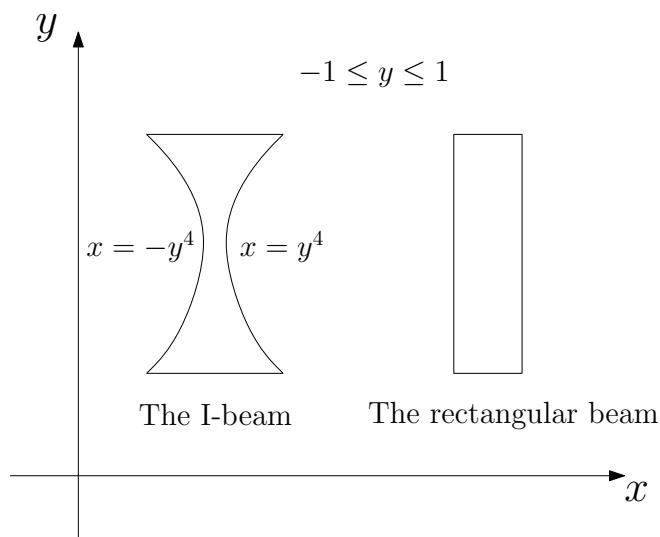


Figure 71: The I-beam and the rectangular beam

Let us compare the stiffness of the I-beam with a rectangular beam as shown in Figure 71. The stiffness (resistance to bending) is proportional to the moment of inertia about the horizontal axis going through the center of mass. For the I-beam, we can compute this as

$$\begin{aligned}
 I_x &= \iint_R \delta(x, y) y^2 dx dy \\
 &= \delta \int_{-1}^1 \left[\int_{-y^4}^{y^4} y^2 dx \right] dy \\
 &= \delta \int_{-1}^1 [xy^2]_{-y^4}^{y^4} dy \\
 &= \delta \int_{-1}^1 (y^6 + y^6) dy \\
 &= 2\delta \frac{y^7}{7} \Big|_{-1}^1 \\
 &= \frac{4}{7}\delta
 \end{aligned}$$

We can compute the area of the I-beam as

$$\begin{aligned} \int_{-1}^1 \left[\int_{-y^4}^{y^4} dx \right] dy &= \int_{-1}^1 [x]_{-y^4}^{y^4} dy \\ &= \int_{-1}^1 (2y^4) dy \\ &= 2 \frac{y^5}{5} \Big|_{-1}^1 \\ &= \frac{4}{5} \end{aligned}$$

We choose a rectangular beam with the same area, so that its width is $\frac{2}{5}$. The moment of inertia of this rectangular beam is

$$\begin{aligned} I_x &= \iint_R \delta(x, y) y^2 dx dy \\ &= \delta \int_{-1}^1 \left[\int_{-\frac{1}{5}}^{\frac{1}{5}} y^2 dx \right] dy \\ &= \delta \int_{-1}^1 [xy^2]_{-\frac{1}{5}}^{\frac{1}{5}} dy \\ &= \delta \int_{-1}^1 \left(\frac{2}{5} y^2 \right) dy \\ &= \frac{2}{15} \delta y^3 \Big|_{-1}^1 \\ &= \frac{4}{15} \delta \end{aligned}$$

This shows that the stiffness of the I-beam is greater than a simple rectangular beam.

14.3.3 Radius of gyration

Let's say a body having moment of inertia I is rotating around a point with angular velocity ω as shown in Figure 69. We want to concentrate all the mass at a single point while having the same I . The distance of this concentrated mass from the center of rotation is the **radius of gyration** r of the body. If the mass of the body is M , then we want $I = Mr^2$, and therefore:

$$r = \sqrt{\frac{I}{M}}$$

Similarly, there is a radius of gyration associated with the moments of inertia I_x and I_y :

$$r_x = \sqrt{\frac{I_x}{M}}, \quad r_y = \sqrt{\frac{I_y}{M}}.$$

14.3.4 Average value

The average value of a function $f(x, y)$ over a region R can be computed using double integrals as

$$\bar{f} = \frac{\iint_R f(x, y) dx dy}{\iint_R dx dy}$$

14.3.5 Application: Design of downhill race-car wheels

We want to design the wheel of a downhill race-car to obtain the fastest speed. The situation is shown in Figure 72.

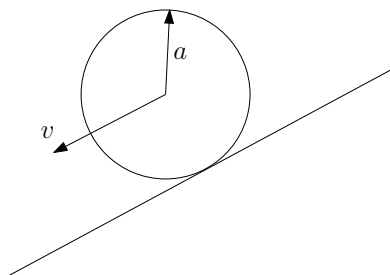


Figure 72: Designing the wheel of a downhill race-car

First, let us look at the kinetic energy KE of the wheel which has two components: the kinetic energy of translation KE_T and the kinetic energy of rotation KE_R . Thus, we have

$$\begin{aligned} KE &= KE_T + KE_R \\ &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \end{aligned}$$

Since, the speed $v = a\omega$, we have

$$\begin{aligned} KE &= \frac{1}{2}Mv^2 + \frac{1}{2}I\frac{v^2}{a^2} \\ &= \frac{1}{2}Mv^2 \left(1 + \frac{I}{Ma^2} \right) \\ &= \frac{1}{2}Mv^2 (1 + I^*) \end{aligned}$$

where $I^* = \frac{I}{Ma^2}$. Also, we note that the kinetic energy at the bottom of the hill is the same as the potential energy at the top of the hill and hence we have

$$\begin{aligned} Mgh &= \frac{1}{2}Mv^2 (1 + I^*) \\ v &= \sqrt{\frac{2gh}{1 + I^*}} \end{aligned}$$

Thus, to go fast, $I^* = I/(Ma^2)$ should be as small as possible. Note that the definition of I gives us

$$I = \iint_R \delta(x, y) r^2 dx dy.$$

This tells us that we want as much mass to be concentrated near the center as possible so that a is large and I small. Therefore, a wheel in the shape of a solid disk is better than a hollow disk. Indeed, for the hollow disk, we get $I^* = 1$ and for the solid disk, we get $I^* = \frac{1}{2}$. For the hollow sphere design, we get $I^* = \frac{2}{3}$; finally, the solid sphere gives us the best design with $I^* = \frac{2}{5}$.

14.3.6 Computing integrals using MATLAB

The symbolic toolbox in MATLAB can be very useful to compute integrals. This is illustrated by a few examples below. The command `int` performs symbolic integrations while `quad` does a numerical integration.

Example 55. Calculate numerically using MATLAB the following single and double integrals:

$$\int_0^1 e^{-2x} dx, \quad \int_0^1 \int_0^1 x \cos(-xy) dy dx$$

$$\int_1^3 \int_1^x \frac{1}{xy} dy dx, \quad \int_0^2 \int_{2y}^4 \frac{y}{(xy)^2 + 1} dx dy$$

1.

$$\int_0^1 e^{-2x} dx$$

The MATLAB code is given by:

```
>>% Compute the integral analytically using symbolic integration
>> syms x
>> int(exp(-2*x),0,1)
ans =
1/2-1/2*exp(-2)
>> double(ans)
ans =
0.432332358381694
>>% Now, we integrate numerically
>> quad(@(x) exp(-2*x),0,1)
ans =
0.432332362588302
```

General syntax:

`int(f(x),xmin,xmax)` where `f(x)` is the function to be integrated and `xmin` and `xmax` are the lower and upper bounds.

`quad(@(x) fun(x),xmin,xmax)` where `fun(x)` should accept a vector argument `x` and return a vector result of values of the integrand.

2.

$$\int_0^1 \int_0^1 x \cos(-xy) dy dx$$

The MATLAB code is given by:

```
>>% Compute the integral analytically using symbolic integration
>> syms x y
>>% First integrate w.r.t. y
>> integral_y = int(x*cos(-x*y),y,0,1)
integral_y =
sin(x)
```

```

>>% Then integrate w.r.t. x
>> int(integral_y,x,0,1)
ans =
1-cos(1)
>> double(ans)
ans =
0.459697694131860
>>% Now, we integrate numerically
>> dblquad(@(x,y) x.*cos(-x*y),0,1,0,1)
ans =
0.459697692948650

```

General syntax: `dblquad(@(x,y) fun(x,y),xmin,xmax,ymin,ymax)` where `fun(x,y)` should accept a vector `x` and a scalar `y` and return a vector result of values of the integrand.

3.

$$\int_1^3 \int_1^x \frac{1}{xy} dy dx$$

The MATLAB code is given by:

```

>>% Compute the integral analytically using symbolic integration
>> syms x positive % We specify that x is a positive variable
>> syms y
>>% First integrate w.r.t. y
>> int(1/(x.*y),y,1,x)
integral_y =
log(x)/x
>>% Then integrate w.r.t. x
>> int(integral_y,1,3)
ans =
1/2*log(3)^2
>> double(ans)
ans =
0.603474480406291
>>% Now, we integrate numerically
>> dblquad(@(x,y) 1./(x*y).*(y<=x),1,3,1,3)
ans =
0.603495789837198

```

Note the use of `(y<=x)` corresponding to the upper bound for the integral over `y`.

4.

$$\int_0^2 \int_{2y}^4 \frac{y}{(xy)^2 + 1} dx dy$$

The MATLAB code is given by:

```

>>% Compute the integral analytically using symbolic integration
>> syms x y
>>% First integrate w.r.t. x
>> integral_x = int(y/((x*y)^2+1),x,2*y,4)
integral_x =
-atan(2*y^2)+atan(4*y)
>>% Then integrate w.r.t. y
>> int(integral_x,y,0,2)
ans =
1/2*atan(5)+1/2*atan(3)-3/8*log(13)+1/8*log(5)
>> double(ans)
ans =
0.550546999677821
>>% Now, we integrate numerically
>> dblquad(@(x,y) y./((x*y).^2+1).*(2*y<=x),0,4,0,2)
ans =
0.550567998152465

```

Reading: Thomas - 15.3, 15.4, 15.5, 15.6

15 Double integrals in Polar Coordinates

Polar coordinates are very useful when there is a polar symmetry in the problem, e.g., the domain of integration. We can define the polar coordinates of a point whose Cartesian coordinates are (x, y) as follows (see Figure 73):

$$x = r \cos \theta$$

$$y = r \sin \theta$$

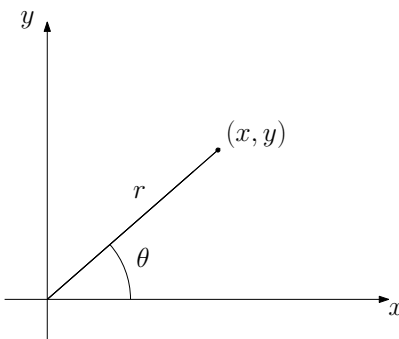


Figure 73: Polar coordinates of a point

Now, we want to convert a function $f(x, y)$ to a function of the polar coordinates $f(r, \theta)$ and obtain the integral $\iint_R f(x, y) dx dy$ in terms of r and θ .

Let us consider a small area of the region of integration, shown as shaded in Figure 74. As shown in the figure, we can calculate this area as $\Delta A = r \Delta r \Delta \theta$. Taking the limit as Δr and $\Delta \theta$ go to

zero, we obtain:

$$\iint_R f(x, y) dx dy = \iint_R f(r, \theta) r dr d\theta$$

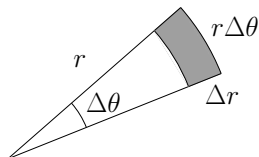


Figure 74: Converting Cartesian coordinates to polar coordinates

Example 56. Find the center of mass of a semi-circular disk bounded by the x -axis and $y = \sqrt{R^2 - x^2}$ using polar coordinates (shown in Figure 75).

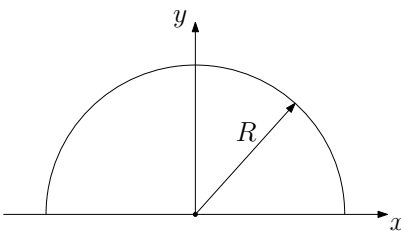


Figure 75: The semi-circular disk

The mass of the semi-circular disk is given by its area, i.e., $m = \frac{\pi R^2}{2}$. Hence, we can compute the center of mass using the equations:

$$\begin{aligned}\bar{x} &= \frac{1}{m} \iint x dx dy \\ \bar{y} &= \frac{1}{m} \iint y dx dy\end{aligned}$$

As the disc is symmetric, we immediately have that $\bar{x} = 0$. By inspection we see that \bar{y} is between 0 and $R/2$. Let's calculate its value now:

$$\begin{aligned}\iint y dx dy &= \int_0^\pi \left[\int_0^R (r \sin \theta) r dr \right] d\theta \\ &= \left(\int_0^\pi \sin \theta d\theta \right) \left(\int_0^R r^2 dr \right) \\ &= \frac{2}{3} R^3\end{aligned}$$

Since $m = \frac{\pi R^2}{2}$, we have $\bar{y} = \frac{4R}{3\pi}$. You can re-do this problem in Cartesian coordinates to check that you get the same values.

Example 57. Find the area of the region R bounded by the circle of radius 1 and the limaçon $r = 2 + \cos \theta$. This is shown in Figure 76.

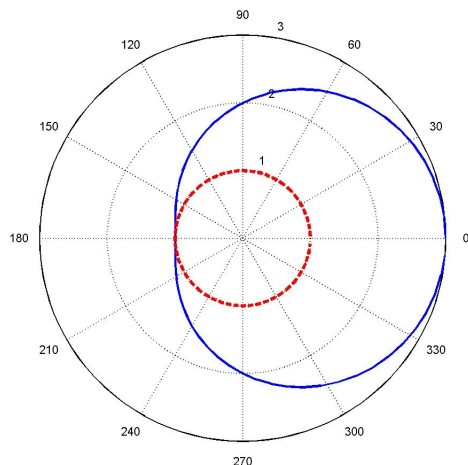


Figure 76: The circle is shown in red (dash) and the limaçon is shown in blue (solid). Region R lies between the circle and the limaçon.

The area can be computed as follows:

$$\begin{aligned}
 A &= \int_{-\pi}^{\pi} \left[\int_1^{2+\cos\theta} r \, dr \right] d\theta \\
 &= \int_{-\pi}^{\pi} \left[\frac{r^2}{2} \right]_1^{2+\cos\theta} d\theta = \int_{-\pi}^{\pi} \left(\frac{(2+\cos\theta)^2}{2} - \frac{1}{2} \right) d\theta \\
 &= \int_{-\pi}^{\pi} \left(\frac{4+4\cos\theta+\cos^2\theta}{2} - \frac{1}{2} \right) d\theta \\
 &= \int_{-\pi}^{\pi} \frac{3}{2} d\theta + \int_{-\pi}^{\pi} 2\cos\theta d\theta + \frac{1}{2} \int_{-\pi}^{\pi} \cos^2\theta d\theta \\
 &= (2\pi) \frac{3}{2} + 2\sin\theta \Big|_{-\pi}^{\pi} + \frac{1}{2} \int_{-\pi}^{\pi} \frac{1+\cos(2\theta)}{2} d\theta \quad \left(\cos^2\theta = \frac{1+\cos(2\theta)}{2} \right) \\
 &= \frac{6\pi}{2} + (0-0) + \frac{1}{4}(2\pi) + \frac{1}{8} \sin(2\theta) \Big|_{-\pi}^{\pi} \\
 &= \frac{7\pi}{2}
 \end{aligned}$$

Example 58. Calculate $\int_0^{\infty} e^{-x^2} dx$ using polar coordinates.

Let us first compute the following integral:

$$\begin{aligned}
 \int_0^{\infty} \int_0^{\infty} e^{-x^2-y^2} dx dy &= \int_0^{\infty} \left[\int_0^{\infty} e^{-x^2} e^{-y^2} dx \right] dy \\
 &= \int_0^{\infty} e^{-y^2} \left[\int_0^{\infty} e^{-x^2} dx \right] dy \\
 &= \left(\int_0^{\infty} e^{-x^2} dx \right) \left(\int_0^{\infty} e^{-y^2} dy \right) \\
 &= \left(\int_0^{\infty} e^{-x^2} dx \right)^2
 \end{aligned}$$

It's equal to the square of the integral we are trying to calculate! We can write the above integral in terms of polar coordinates to get:

$$\begin{aligned} \int_0^\infty \int_0^\infty e^{-x^2-y^2} dx dy &= \int_0^{\frac{\pi}{2}} \left[\int_0^\infty e^{-r^2} r dr \right] d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[\frac{e^{-r^2}}{-2} \right]_0^\infty d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta \\ &= \frac{\pi}{4} \end{aligned}$$

Hence, we have

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Example 59. Find the volume of the solid region which is given by the intersection of a sphere described by $x^2 + y^2 + z^2 = 4$ and a cylinder described by $(x-1)^2 + y^2 = 1$. See Figure 77.

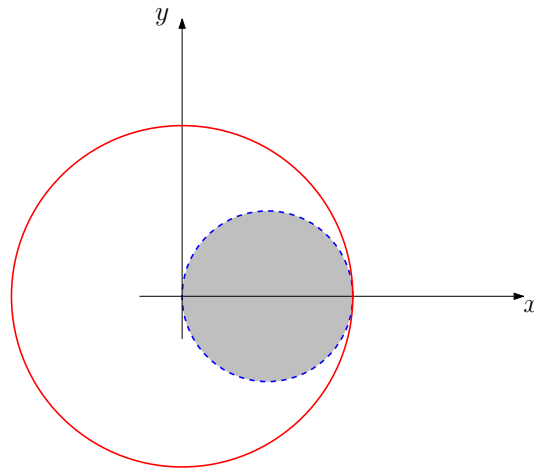


Figure 77: Top view of the intersection region (shaded). Boundary for the sphere is shown in red (solid) and for the cylinder in blue (dashed).

We can compute the volume of the region as the integral:

$$V = \iint_R 2\sqrt{4-x^2-y^2} dx dy$$

Also, we can write x and y in terms of the polar coordinates as $x = r \cos \theta$ and $y = r \sin \theta$. The equation for the boundary of cylinder can be written in polar coordinates as:

$$\begin{aligned} (x-1)^2 + y^2 &= 1 \\ \Rightarrow (r \cos \theta - 1)^2 + (r \sin \theta)^2 &= 1 \\ \Rightarrow r - 2 \cos \theta &= 0 \\ \Rightarrow r &= 2 \cos \theta \end{aligned}$$

The final calculation is difficult. You can use Matlab to find the answer

```

>> syms r positive
>> syms theta
>> int(4*r*(4-r^2)^(1/2),r,0,2*cos(theta))
ans =
32/3+32/3*(1-cos(theta)^2)^(1/2)*cos(theta)^2-32/3*(1-cos(theta)^2)^(1/2)
>> int(ans,theta,0,pi/2)
ans =
-64/9+16/3*pi

```

or follow these steps:

$$\begin{aligned}
 V &= 4 \int_0^{\pi/2} \int_0^{2\cos\theta} (\sqrt{4-r^2})r \, dr \, d\theta = 4 \int_0^{\pi/2} \left[\int_0^{2\cos\theta} (\sqrt{4-r^2})r \, dr \right] d\theta \\
 &= 4 \int_0^{\pi/2} \left[-\frac{1}{3}(4-r^2)^{3/2} \right]_0^{2\cos\theta} d\theta = \dots
 \end{aligned}$$

$$\begin{aligned}
 V &= 4 \int_0^{\pi/2} \left(-\frac{8}{3}(1-\cos^2\theta)^{3/2} + \frac{8}{3} \right) d\theta = \frac{32}{3} \int_0^{\pi/2} (1-\sin^3\theta) d\theta \\
 &= \frac{16}{3} \left(\frac{\pi}{2} - \int_0^{\pi/2} \sin\theta(1-\cos^2\theta)d\theta \right) = \frac{16}{3} \left(\frac{\pi}{2} - \left[-\cos\theta + \frac{\cos^3\theta}{3} \right]_0^{\pi/2} \right) \\
 &= \frac{16}{3} \left(\frac{\pi}{2} - \left(-1 + \frac{1}{3} \right) \right) = \frac{16\pi}{3} - \frac{64}{9} \\
 &\approx 9.6
 \end{aligned}$$

16 Triple integrals

We can define a triple integral as follows:

$$\iiint_D f(x,y,z) \, dx \, dy \, dz = \lim_{h \rightarrow 0} h^3 \sum_i f(x_i, y_i, z_i)$$

The sum on the right hand side is analogous to the one used to defined 2D integrals, the difference being that we now have a 3D volume D instead of a 2D region R . Each volume element can be chosen to be a small cube with volume h^3 .

16.1 Integration over a parallelepiped

16.1.1 Fubini's Theorem

The triple integral can be written as:

$$\iiint_D f(x,y,z) \, dx \, dy \, dz = \int_a^b \left[\int_c^d \left[\int_e^f f(x,y,z) \, dz \right] dy \right] dx$$

Note that the order of integration does not matter and we can permute the order of the variables x , y and z .

16.2 Integration over a general region

We define a z -simple region (in 3D) as shown in Figure 78 where the z coordinate is bounded by 2 surfaces z_1, z_2 and we have $z_1(x, y) \leq z \leq z_2(x, y)$.

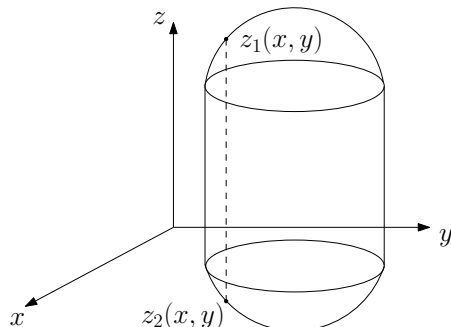


Figure 78: A z -simple volume

If we have a z -simple region, we can write the triple integral as:

$$\iiint_D f(x, y, z) dx dy dz = \iint_R \left[\int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \right] dx dy$$

R is the vertical projection of the volume D onto the x - y plane. The integral over R can be obtained using the methods for 2D integrals.

Similarly, we can define the triple integrals over x -simple and y -simple regions in 3D.

The volume of a domain D is given by

$$V = \iiint_D dx dy dz$$

Example 60. Using triple integrals, find the volume of the region enclosed by the plane $x + y + z = 1$ and the three coordinate planes. The plane is shown in Figure 79.

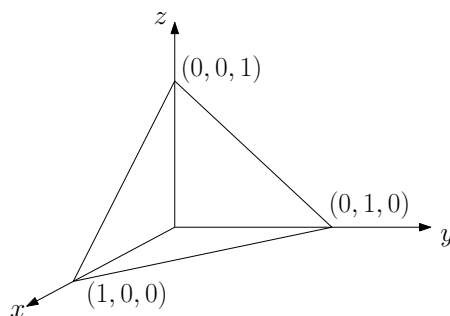


Figure 79: The plane $x + y + z = 1$

We can see that the region is z -simple for $0 \leq z \leq 1 - x - y$. The projected region in the x - y plane

is y -simple and hence we can write the volume as the following iterated integral:

$$\begin{aligned} V &= \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} dz \right] dy \right] dx \\ &= \int_0^1 \left[\int_0^{1-x} (1-x-y) dy \right] dx \\ &= \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2} \right) dx = \frac{1}{6} \end{aligned}$$

16.3 Applications of triple integrals

16.3.1 Computing the mass

Let's say we have the density $\delta(x, y, z)$ function over an object and want to compute its mass. Recall that density gives us mass per unit volume, and hence the mass can be computed as

$$M = \iiint_D \delta(x, y, z) dx dy dz$$

16.3.2 Centroid/Center of mass

The location of the center of mass for a 3D object is given by:

$$\begin{aligned} \bar{x} &= \frac{1}{M} \iiint_D x \delta(x, y, z) dx dy dz \\ \bar{y} &= \frac{1}{M} \iiint_D y \delta(x, y, z) dx dy dz \\ \bar{z} &= \frac{1}{M} \iiint_D z \delta(x, y, z) dx dy dz \end{aligned}$$

16.3.3 Moment of Inertia

We can define the moment of inertia of a 3D object about the three coordinate axis as follows:

$$\begin{aligned} I_x &= \iiint_D (y^2 + z^2) \delta(x, y, z) dx dy dz \\ I_y &= \iiint_D (x^2 + z^2) \delta(x, y, z) dx dy dz \\ I_z &= \iiint_D (x^2 + y^2) \delta(x, y, z) dx dy dz \end{aligned}$$

16.3.4 Radius of gyration

The radius of gyration associated with the three moments of inertia are given as follows:

$$r_x = \sqrt{\frac{I_x}{M}}, \quad r_y = \sqrt{\frac{I_y}{M}}, \quad r_z = \sqrt{\frac{I_z}{M}}.$$

16.3.5 Average value

The average value of a function $f(x, y, z)$ over a volume D can be computed using triple integrals:

$$\bar{f} = \frac{\iiint_D f(x, y, z) dx dy dz}{\iiint_D dx dy dz}$$

Example 61. Find the center of mass of a solid of constant density δ bounded by the disk $x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $0 \leq z \leq 4 - x^2 - y^2$ (see Figure 80).

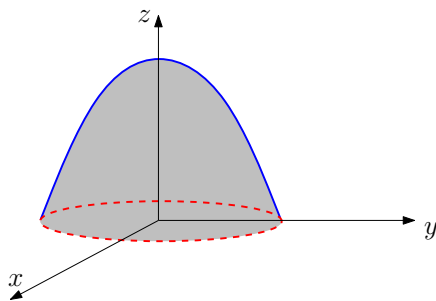


Figure 80: The solid volume is shown as shaded. The disk boundary is shown in red (dashed) and the paraboloid is shown in blue (solid)

This volume is z -simple and hence we can compute the mass as follows:

$$\begin{aligned} M &= \iiint_D dx dy dz \\ &= \iint_R \left[\int_0^{4-x^2-y^2} dz \right] dx dy \\ &= \iint_R (4 - x^2 - y^2) dx dy \end{aligned}$$

Now, we can switch to polar coordinates to get:

$$\begin{aligned} M &= \iint_R (4 - x^2 - y^2) dx dy \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta = 8\pi \end{aligned}$$

We can see that $\bar{x} = \bar{y} = 0$. Let's now calculate the integral for \bar{z} :

$$\begin{aligned} \iiint_D z dx dy dz &= \iint_R \left[\int_0^{4-x^2-y^2} z dz \right] dx dy \\ &= \frac{1}{2} \iint_R (4 - x^2 - y^2)^2 dx dy \end{aligned}$$

Again, we can switch to polar coordinates to get:

$$\iiint_D z dx dy dz = \frac{1}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2 r dr d\theta = \frac{32\pi}{3}$$

Hence we get $\bar{z} = 4/3$.

Example 62. Find the mass and moments of inertia I_x , I_y and I_z for the rectangular solid of constant density δ and dimensions in the x , y and z directions of a , b and c respectively. The origin $(0,0,0)$ is at the center of the solid.

We can compute

$$\begin{aligned} I_x &= \delta \int_{-\frac{a}{2}}^{\frac{a}{2}} \left[\int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\int_{-\frac{c}{2}}^{\frac{c}{2}} (y^2 + z^2) dz \right] dy \right] dx \\ &= \delta \int_{-\frac{a}{2}}^{\frac{a}{2}} \left[\int_{-\frac{b}{2}}^{\frac{b}{2}} c \left(y^2 + \frac{1}{12} c^2 \right) dy \right] dx \\ &= \delta \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{b}{12} (b^2 + c^2) dx = \frac{abc\delta}{12} (b^2 + c^2) \end{aligned}$$

Note that the mass of the solid is given by $M = abc\delta$ and hence we have

$$I_x = \frac{M}{12} (b^2 + c^2)$$

We can similarly compute:

$$I_y = \frac{M}{12} (a^2 + c^2), \quad I_z = \frac{M}{12} (a^2 + b^2).$$

Example 63. Find the volume of the oblique segment bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = y + 2$. The shape of the volume is depicted in Figure 81(a).

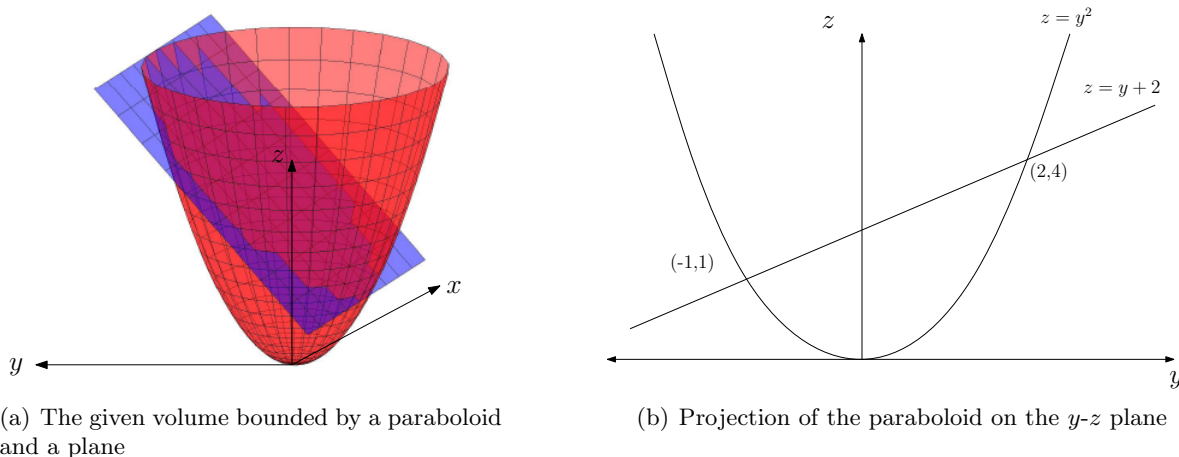


Figure 81: Computing the volume of the given oblique segment

We can see that the region is x -simple and the bounds for x are given by: $-\sqrt{z - y^2} \leq x \leq \sqrt{z - y^2}$. If we project the surface onto the y - z plane, we obtain a 2D region bounded by the parabola $y = z^2$ and the line $z = y + 2$ as shown in Figure 81(b). This region is z -simple and we can integrate z in the region $y^2 \leq z \leq y + 2$. To obtain the bounds for y , we can solve for the points of intersection of the parabola and the line using the equation $y^2 = y + 2$: the bounds are $-1 \leq y \leq 2$.

Now that we have the bounds for all the integrals, we can compute the volume as:

$$\begin{aligned} V &= \int_{-1}^2 \left[\int_{y^2}^{y+2} \left[\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx \right] dz \right] dy \\ &= \int_{-1}^2 \left[\int_{y^2}^{y+2} 2\sqrt{z-y^2} dz \right] dy = \int_{-1}^2 \left[\frac{4}{3}(z-y^2)^{3/2} \right]_{y^2}^{y+2} dy \\ &= \int_{-1}^2 \frac{4}{3}(y+2-y^2)^{3/2} dy = \int_{-1}^2 \frac{4}{3}(y+2-y^2)^{3/2} dy \end{aligned}$$

The calculation of this integral is a long process. If we decide to use Matlab the code is:

```
>> syms x y
>> syms z positive
>> int(1,x,-(z-y^2)^(1/2),(z-y^2)^(1/2))
ans =
2*(z-y^2)^(1/2)
>> int(ans,z,y^2,y+2)
ans =
4/3*(y+2-y^2)^(3/2)
>> int(ans,y,-1,2)
ans =
81/32*pi
```

By hand, we first need to complete the square for $y+2-y^2$:

$$y+2-y^2 = -\left(y-\frac{1}{2}\right)^2 + \frac{9}{4}$$

This leads to the substitution $u = y - \frac{1}{2}$ in the integral:

$$\begin{aligned} V &= \frac{4}{3} \int_{-\frac{3}{2}}^{\frac{3}{2}} \left(\frac{9}{4} - u^2\right)^{3/2} du \\ &= \frac{4}{3} \int_{-\frac{3}{2}}^{\frac{3}{2}} \left(\frac{9}{4}\right)^{3/2} \left(1 - \left(\frac{2u}{3}\right)^2\right)^{3/2} du \end{aligned}$$

We can further substitute $\frac{2u}{3} = \cos \theta$ in the integral to get:

$$\begin{aligned} V &= \frac{4}{3} \int_{\pi}^0 \left(\frac{9}{4}\right)^{3/2} (1 - \cos^2 \theta)^{3/2} \frac{3}{2} (-\sin \theta) d\theta \\ &= \frac{4}{3} \left(\frac{3^4}{2^4}\right) \int_0^{\pi} (\sin^2 \theta)^{3/2} (\sin \theta) d\theta \\ &= \frac{27}{4} \int_0^{\pi} \sin^4 \theta d\theta = \frac{27}{4} \left(\frac{3\pi}{2^3}\right) = \frac{81\pi}{32} \end{aligned}$$

16.4 Triple integrals in cylindrical coordinates

Many problems encountered in engineering have a cylindrical symmetry, such as the flow of a fluid in a pipe, the temperature in a circular rod, etc. As example, a cylinder is defined by $r = 1$, a paraboloid by $z = r^2$ and a cone by $z = r$.

The Cartesian coordinates (x, y, z) are written using cylindrical coordinates (r, θ, z) as:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

z is unchanged of course.

For transforming an integral from Cartesian coordinates to cylindrical coordinates, we can compute the volume element in cylindrical coordinates. It is equal to $r dr d\theta dz$; it is a simple extension of the case of polar coordinates.

Example 64. Using cylindrical coordinates find the volume and the center of mass of a cone of uniform density formed by $z = \sqrt{x^2 + y^2}$ and $z = 1$.

We can compute the volume as:

$$\begin{aligned} V &= \int_0^1 \left[\int_0^{2\pi} \left[\int_r^1 dz \right] d\theta \right] r dr \\ &= \int_0^1 \left[\int_0^{2\pi} (1-r) d\theta \right] r dr \\ &= \int_0^1 2\pi(1-r)r dr = 2\pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{3} \end{aligned}$$

By symmetry, we have $\bar{x} = \bar{y} = 0$; \bar{z} satisfies the following bounds: $\frac{1}{2} < \bar{z} < 1$. We now compute \bar{z} :

$$\begin{aligned} \iiint_D z \delta(x, y, z) dx dy dz &= \int_0^{2\pi} \left[\int_0^1 \left[\int_r^1 z dz \right] r dr \right] d\theta \\ &= \int_0^{2\pi} \left[\int_0^1 \frac{1}{2} (1-r^2) r dr \right] d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4} \end{aligned}$$

Thus, we have $\bar{z} = \frac{3}{\pi} \frac{\pi}{4} = \frac{3}{4}$.

Example 65. Using cylindrical coordinates find the volume above the x - y plane bounded by $x^2 + y^2 = 1$ and $x + y + z = 2$.

First, we have to determine the bounds of integration. For this, we need to calculate the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 2$. The points on the intersection satisfy:

$$x^2 + y^2 = 1, \quad x + y + z = 2$$

Using cylindrical coordinates, these points can be parametrized by:

$$x = \cos \theta, \quad y = \sin \theta, \quad z = 2 - \cos \theta - \sin \theta$$

with $0 \leq \theta \leq 2\pi$. In particular, we see that z is always positive. Therefore we have a z -simple region with the bounds: $0 \leq z \leq 2 - x - y$, $0 \leq x^2 + y^2 \leq 1$.

Second, we calculate the volume using cylindrical coordinates:

$$\begin{aligned}
 V &= \iint \left[\int_0^{2-x-y} dz \right] r dr d\theta \\
 &= \iint (2-x-y) r dr d\theta \\
 &= \int_0^{2\pi} \left[\int_0^1 r(2-r\cos\theta-r\sin\theta) dr \right] d\theta \\
 &= \int_0^{2\pi} \left[\int_0^1 2r dr \right] d\theta = 4\pi \int_0^1 r dr = 2\pi
 \end{aligned}$$

16.5 Triple integrals in spherical coordinates

We define the spherical coordinates as (r, θ, ϕ) , where: r is the distance to the origin, ϕ is the angle with z -axis and θ is the angle made by the projection on the x - y plane with the x -axis, as shown in Figure 82. Thus, a point having the spherical coordinates (r, θ, ϕ) has the Cartesian coordinates:

$$\begin{aligned}
 x &= r \sin \phi \cos \theta, \\
 y &= r \sin \phi \sin \theta, \\
 z &= r \cos \phi.
 \end{aligned}$$

To convert integrals from Cartesian coordinates to spherical coordinates, we calculate the volume element as before. It is shown in Figure 83; we can write the volume element as:

$$\Delta V = (\Delta r)(r\Delta\phi)(r \sin \phi\Delta\theta) = r^2 \sin \phi \Delta r \Delta \phi \Delta \theta.$$

Example 66. Using spherical coordinates find the center of mass of a hemisphere of radius R .

The hemisphere fixes the range of the spherical coordinates as: $0 \leq r \leq R$, $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$. We can see from symmetry that $\bar{x} = \bar{y} = 0$. Also, we know that the volume of the hemisphere is given by $V = \frac{1}{2} \frac{4}{3} \pi R^3 = \frac{2\pi}{3} R^3$. To compute \bar{z} , we need to compute the integral

$$\begin{aligned}
 \iiint_D z \delta(x, y, z) dx dy dz &= \int_0^{2\pi} \left[\int_0^{\frac{\pi}{2}} \left[\int_0^R z r^2 \sin \phi dr \right] d\phi \right] d\theta \\
 &= \int_0^{2\pi} \left[\int_0^{\frac{\pi}{2}} \left[\int_0^R r^3 \sin \phi \cos \phi dr \right] d\phi \right] d\theta \\
 &= \int_0^{2\pi} \left[\int_0^{\frac{\pi}{2}} \frac{R^4}{4} \sin \phi \cos \phi d\phi \right] d\theta \\
 &= \frac{R^4}{8} \int_0^{2\pi} \left[\int_0^{\frac{\pi}{2}} \sin(2\phi) d\phi \right] d\theta \\
 &= \frac{R^4}{8} \int_0^{2\pi} d\theta = \frac{\pi R^4}{4}
 \end{aligned}$$

The Matlab code for this calculation is:

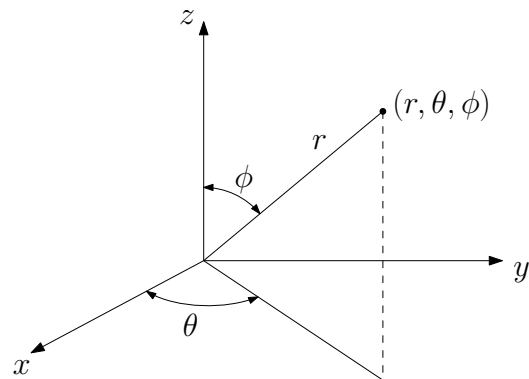


Figure 82: Definition of the spherical coordinate system.

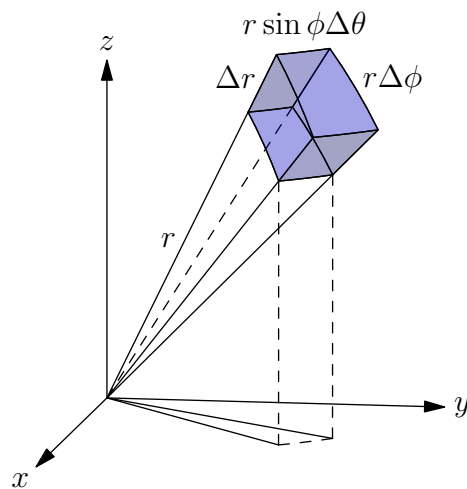


Figure 83: A volume element in spherical coordinates.

```

>> syms R positive
>> syms r positive
>> syms phi positive
>> syms theta
>> z = r*cos(phi);
>> int(z*r^2*sin(phi),r,0,R)
ans =
1/4*cos(phi)*sin(phi)*R^4
>> int(ans,phi,0,pi/2)
ans =
1/8*R^4
>> int(ans,theta,0,2*pi)
ans =
1/4*R^4*pi

```

Example 67. Find the moment of inertia of a uniform solid sphere of mass M and radius R with respect to an axis going through the origin.

Note that the direction of the axis is not mentioned, but it is immaterial as the sphere is isotropic. The equation of the sphere gives us the following relationship between the Cartesian and spherical coordinates:

$$x^2 + y^2 = r^2 \sin^2 \phi$$

Now, we can compute the moment of inertia about the z -axis (remember that it is same for all directions) as:

$$\begin{aligned}
 I_z &= \int_0^{2\pi} \left[\int_0^\pi \left[\int_0^R r^4 \sin^3 \phi dr \right] d\phi \right] d\theta \\
 &= \int_0^{2\pi} \left[\int_0^\pi \left(\frac{R^5}{5} \right) \sin^3 \phi d\phi \right] d\theta \\
 &= \frac{R^5}{5} \int_0^{2\pi} \left[\int_0^\pi \sin^3 \phi d\phi \right] d\theta \\
 &= \frac{R^5}{5} \int_0^{2\pi} \frac{4}{3} d\theta = 2\pi \left(\frac{R^5}{5} \right) \left(\frac{4}{3} \right) = \frac{8\pi}{15} R^5
 \end{aligned}$$

Since we know that the mass of the sphere is given by $M = \frac{4}{3}\pi R^3$, we can write the moment of inertia for any direction as:

$$I = \frac{2}{5}MR^2$$

The Matlab code is:

```

>> syms R positive
>> syms r positive
>> syms phi positive
>> syms theta
>> int(r^2*sin(phi)^2*r^2*sin(phi),r,0,R)

```

```

ans =
1/5*sin(phi)^3*R^5
>> int(ans,phi,0,pi)
ans =
4/15*R^5
>> int(ans,theta,0,2*pi)
ans =
8/15*R^5*pi

```

Reading: Thomas - 15.7, 16.1, 16.2

17 Transformation of variables

Recall that we can do a change of variables in a 1D integral where x is a function of u as:

$$\int_a^b f(x) dx = \int_c^d f(x(u)) x'(u) du$$

The reason that we would want to do a change of variables is that the transformed integral might be easier to calculate.

Example 68. *If we want to integrate around a circle, then the coordinates are given by $x = r \cos \theta$ and $y = r \sin \theta$. It might be easier to integrate after transforming to polar coordinates (r, θ) as the region of integration is rectangular as shown in Figure 84.*

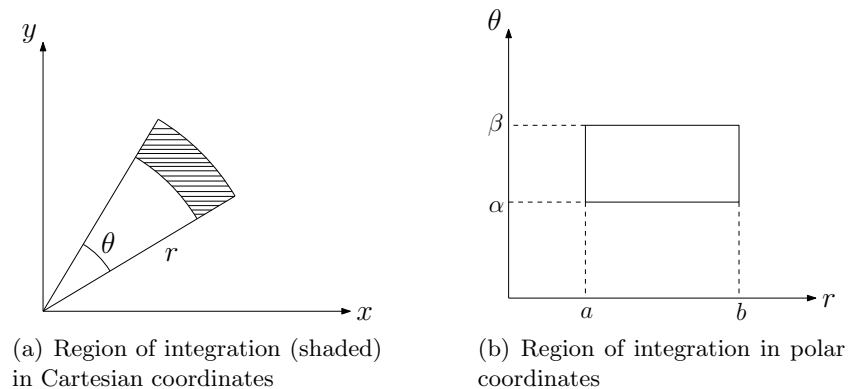


Figure 84: Change in region of integration after coordinate transformation

Example 69. *Suppose we want to integrate over the two hyperbolas: $u = xy$ and $v = x^2 - y^2$. The figure shows that the region of integration is much simpler in the u - v coordinates and hence we might want to switch to these coordinates. We can convert an integral in the x - y plane to an integral in the u - v plane by equating an area element in both the planes. The area element is shown in Figure 85.*

We can linearize the region in the x - y plane to get a parallelogram as shown in Figure 86.

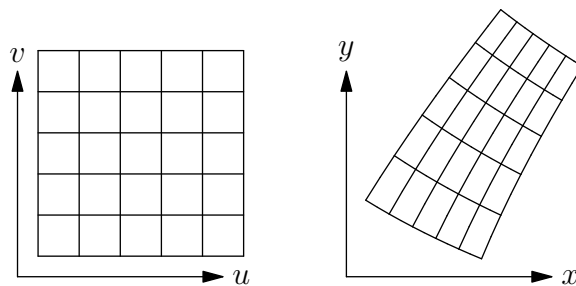


Figure 85: Change in region of integration after coordinate transformation; $u = xy$ and $v = x^2 - y^2$.

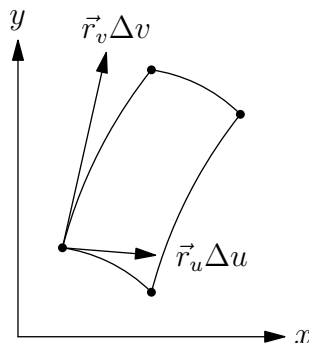


Figure 86: Approximating the region of integration by a parallelogram.

We can now compute the area of the parallelogram as

$$\begin{aligned}
 \Delta A &= \left| \left(\Delta u \frac{\partial \vec{r}}{\partial u} \right) \times \left(\Delta v \frac{\partial \vec{r}}{\partial v} \right) \right| \\
 &= \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta u \Delta v \\
 &= \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| \Delta u \Delta v \\
 &= \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right| \Delta u \Delta v
 \end{aligned}$$

as $\vec{r} = (x, y, 0)$, and hence $\frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \right)$, $\frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \right)$. The **Jacobian** of the transformation is defined as:

$$\frac{\partial(x, y)}{\partial(u, v)} \stackrel{\text{def}}{=} \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

Now, we can transform the 2D integral using the formula:

$$\iint f(x, y) dx dy = \iint f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

This is a general formula which can be applied to any coordinate transformation.

Example 70. Find the polar moment of inertia of the plane region bounded by $1 \leq xy \leq 3$ and $1 \leq x^2 - y^2 \leq 4$. The density $\delta(x, y)$ is equal to 1.

The polar moment of inertia is given by

$$I = \iint (x^2 + y^2) dx dy$$

We will transform to the coordinates (u, v) , where:

$$u = xy, \quad v = x^2 - y^2$$

To do this transformation, we need to compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

The matrix

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Therefore, from linear algebra, we have that

$$\frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = 1$$

We compute $\partial(u, v)/\partial(x, y)$ because this is easier:

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2y^2 - 2x^2 = -2(x^2 + y^2)$$

This gives us:

$$dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \frac{1}{\left| \frac{\partial(u, v)}{\partial(x, y)} \right|} du dv = \frac{1}{2(x^2 + y^2)} du dv$$

Don't forget to take the absolute value. Finally, we can compute the moment of inertia as

$$\begin{aligned} I &= \iint (x^2 + y^2) dx dy \\ &= \iint (x^2 + y^2) \frac{1}{2(x^2 + y^2)} du dv \\ &= \frac{1}{2} \iint du dv = \frac{1}{2} \int_1^4 \int_1^3 du dv = 3 \end{aligned}$$

Example 71. Using transformation of coordinates find the area enclosed by $y = x$, $y = 2x$, $y = 1/x$ and $y = 2/x$.

We transform the coordinates to (u, v) , where

$$u = \frac{y}{x}, \quad v = xy$$

As in the previous example, we can compute the Jacobian by first computing

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -\frac{y}{x} - \frac{y}{x} = -2\frac{y}{x} = -2u$$

Hence, we can find the area as

$$\begin{aligned} A &= \iint dx dy \\ &= \iint \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \int_1^2 \int_1^2 \frac{1}{2u} du dv = \frac{1}{2} \ln 2 \end{aligned}$$

Example 72. Find the Jacobian of transformation between Cartesian and polar coordinates.

We have $x = r \cos \theta$ and $y = r \sin \theta$. We can compute the Jacobian of the transformation as

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$$

Hence we can write the transformed integral as:

$$\begin{aligned} \iint f(x, y) dx dy &= \iint f(r, \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta \\ &= \iint f(r, \theta) r dr d\theta \end{aligned}$$

This is the transformation formula we introduced previously!

17.1 Transformations in triple integrals

We can write the Jacobian as

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

We can transform coordinates in triple integrals using

$$\iiint f(x, y, z) dx dy dz = \iiint f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

For example, the transformation to cylindrical coordinates can be computed using:

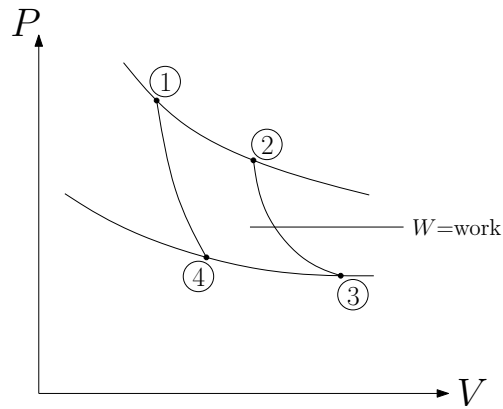
$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| = r$$

The transformation to spherical coordinates can be computed using:

$$\left| \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} \right| = r^2 \sin \phi$$

An important property of the Jacobian matrix is that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)} = 1$$

Figure 87: The ideal carnot cycle P - V diagram

17.2 Application: Carnot cycle

There are 4 steps in the cycle that converts heat to mechanical work (as shown in Figure 87):

1. The first process performed on the gas is an **isothermal expansion**. The heat source is brought into contact with the cylinder. The temperature remains constant while the volume increases. During the process from **State 1** to **State 2** heat is transferred from the source to the gas to maintain the temperature. We will note the heat transfer to the gas by Q_1 . The process is described by the equation:

$$PV = nRT = 3$$

2. The second process performed on the gas is an **adiabatic expansion**. During an adiabatic process no heat is transferred to the gas. The temperature decreases and the volume increases as the gas expands to fill the volume. The process from **State 2** to **State 3** is described by the equation:

$$PV^\gamma = 2$$

3. The third process performed on the gas is an **isothermal compression**. A cold source is brought into contact with the cylinder. The temperature remains constant while the volume decreases. During the process from **State 3** to **State 4** heat is transferred from the gas to the cold source to maintain the temperature. We will note the heat transfer by Q_2 away from the gas. The process is described by the equation:

$$PV = 1$$

4. The fourth process performed on the gas is an **adiabatic compression**. The temperature increases and the volume decreases as the gas is compressed. During the process from **State 4** to **State 1** no heat is transferred. The process is described by the equation:

$$PV^\gamma = 1$$

The work done by the engine is given by

$$W = Q_1 - Q_2$$

which is also given by the area enclosed by the curve plotted in Figure 87. It can be computed using a double integral as $W = \iint dP dV$. We can make the integral easier to calculate by transforming the variables (P, V) to (u, v) , where $u = PV$ and $v = PV^\gamma$. We get:

$$W = \iint \left| \frac{\partial(P, V)}{\partial(u, v)} \right| du dv$$

As before, we compute the inverse of the Jacobian as

$$\frac{\partial(u, v)}{\partial(P, V)} = \begin{vmatrix} \frac{\partial u}{\partial P} & \frac{\partial u}{\partial V} \\ \frac{\partial v}{\partial P} & \frac{\partial v}{\partial V} \end{vmatrix} = \begin{vmatrix} V & P \\ V^\gamma & \gamma PV^{\gamma-1} \end{vmatrix} = \gamma PV^\gamma - PV^\gamma = (\gamma - 1)PV^\gamma = (\gamma - 1)v$$

Thus, we get

$$\frac{\partial(P, V)}{\partial(u, v)} = \frac{1}{(\gamma - 1)v}$$

Hence, work can be calculated as

$$\begin{aligned} W &= \int_1^3 \int_1^2 \frac{1}{(\gamma - 1)v} dv du \\ &= \int_1^3 \frac{\ln 2}{\gamma - 1} du = \frac{2 \ln 2}{\gamma - 1} = 5 \ln 2 \end{aligned}$$

A movie depicting the Carnot cycle is also uploaded at coursework. You should check it out!

18 Line integrals

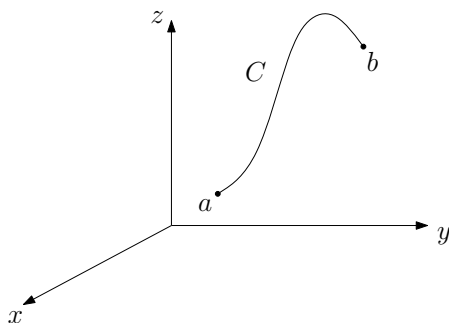


Figure 88: A wire going from a to b along a curve C .

Suppose $f(x, y, z)$ denotes the linear density of a wire (shown in Figure 88) at the point (x, y, z) . Then the mass of the wire is given by

$$M = \int_C f(x, y, z) ds$$

The integral is called a **line integral** with respect to the arc length s . We can make a change of variables to compute this line integral. Recall from the earlier part of class that $ds = |\vec{v}| dt$ and hence we have

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) |\vec{v}(t)| dt$$

Example 73. Calculate $\int_C xy \, ds$ where C is the part of the unit circle that lies in the first quadrant.

We can parameterize the coordinates as $x = \cos t$ and $y = \sin t$. Note that here $|\vec{v}| = 1$. Now, we can compute the integral as

$$\begin{aligned} \int_C xy \, ds &= \int_0^{\frac{\pi}{2}} (\cos t) (\sin t) (1) \, dt \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin(2t) \, dt = \frac{1}{2} \left[\frac{-\cos(2t)}{2} \right]_0^{\frac{\pi}{2}} = \frac{1}{2} \end{aligned}$$

18.1 Applications

18.1.1 Computing the mass

Let's say we have the density $\delta(x, y, z)$ function along a cable. The mass of the cable is given by:

$$M = \int_C \delta(x, y, z) \, ds$$

18.1.2 Centroid/Center of mass

The location of the center of mass for a 3D curve is given by

$$\bar{x} = \frac{1}{M} \int_C x \delta(x, y, z) \, ds$$

$$\bar{y} = \frac{1}{M} \int_C y \delta(x, y, z) \, ds$$

$$\bar{z} = \frac{1}{M} \int_C z \delta(x, y, z) \, ds$$

18.1.3 Moment of Inertia

We can define the moment of inertia of a 3D curve about the three coordinate axis as follows:

$$I_x = \int_C (y^2 + z^2) \delta(x, y, z) \, ds$$

$$I_y = \int_C (x^2 + z^2) \delta(x, y, z) \, ds$$

$$I_z = \int_C (x^2 + y^2) \delta(x, y, z) \, ds$$

Example 74. Find the location of the center of mass of the helix C described by $\vec{r}(t) = 3 \cos t \vec{i} + 3 \sin t \vec{j} + 4t \vec{k}$, whose density is given by $\delta = kz$.

We parameterize the curve as $x = 3 \cos t$, $y = 3 \sin t$ and $z = 4t$. Note that the velocity is given by $\vec{v} = (-3 \sin t, 3 \cos t, 4)$ and hence $|\vec{v}| = 5$. Thus, the total mass is given as

$$\begin{aligned} M &= \int_C kz \, ds \\ &= \int_0^{\pi} (kz) (5) \, dt = 20k \int_0^{\pi} t \, dt = 10k\pi^2 \end{aligned}$$

We can compute the location of the center of mass along x as follows:

$$\begin{aligned}\bar{x} &= \frac{1}{M} \int_C x \delta(x, y, z) ds \\ &= \frac{1}{10k\pi^2} \int_0^\pi (k4t) 3(\cos t) (5) dt \\ &= \frac{60k}{10k\pi^2} \int_0^\pi t \cos t dt\end{aligned}$$

Matlab code:

```
>> syms t
>> syms k positive
>> M = int(20*k*t,t,0,pi)
M =
10*k*pi^2
>> int(3*cos(t)*k*4*t*5,t,0,pi)
ans =
-120*k
>> x = ans/M
x =
-12/pi^2
>> int(3*sin(t)*k*4*t*5,t,0,pi)
ans =
60*k*pi
>> y = ans/M
y =
6/pi
>> int(4*t*k*4*t*5,t,0,pi)
ans =
80/3*k*pi^3
>> z = ans/M
z =
8/3*pi
```

The integral can be computed by hand using:

$$\begin{aligned}\bar{x} &= \frac{60k}{10k\pi^2} \int_0^\pi t \cos t dt = \frac{6}{\pi^2} \left([t \sin t]_0^\pi - \int_0^\pi \sin t dt \right) \quad (\text{Integrating by parts}) \\ &= \frac{6}{\pi^2} (0 + [\cos t]_0^\pi) = -\frac{12}{\pi^2}\end{aligned}$$

We can compute the location of the center of mass along y as follows:

$$\begin{aligned}\bar{y} &= \frac{1}{M} \int_C y \delta(x, y, z) ds = \frac{1}{M} \int_0^\pi (k4t) 3(\sin t) (5) dt \\ &= \frac{60k}{M} \int_0^\pi t \sin t dt \\ &= \frac{60k}{M} \left([-t \cos t]_0^\pi - \int_0^\pi -\cos t dt \right) \quad (\text{Integrating by parts}) \\ &= \frac{60k}{M} (\pi + [\sin t]_0^\pi) = \frac{6}{\pi}\end{aligned}$$

We can compute the location of the center of mass along z as follows:

$$\begin{aligned}\bar{z} &= \frac{1}{M} \int_C z \delta(x, y, z) ds = \frac{1}{M} \int_0^\pi (k4t) (4t) (5) dt \\ &= \frac{80k}{M} \int_0^\pi t^2 dt = \frac{80k}{M} \frac{\pi^3}{3} = \frac{8\pi}{3}\end{aligned}$$

18.2 Line integral with respect to coordinate variables

The line integral with respect to coordinate x is defined as:

$$\int_C f(x, y, z) dx$$

As before, if we have the trajectory vector $\vec{r}(t) = (x(t), y(t), z(t))$ then we have

$$dx = \left(\frac{dx}{dt} \right) dt = x' dt$$

Hence, we can write

$$\int_C f(x, y, z) dx = \int_a^b f(x(t), y(t), z(t)) x'(t) dt$$

Note that there is no absolute value around $x'(t)$. Hence the integral can have any sign even if $f \geq 0$. In particular, the sign will depend on the direction chosen when tracing the curve. See section 18.2.1 below.

Similarly, we can compute

$$\begin{aligned}\int_C f(x, y, z) dy &= \int_a^b f(x(t), y(t), z(t)) y'(t) dt \\ \int_C f(x, y, z) dz &= \int_a^b f(x(t), y(t), z(t)) z'(t) dt\end{aligned}$$

Example 75. Calculate $\int_C y dx + z dy + x dz$, where C is parameterized by: $x = t$, $y = t^2$, $z = t^3$, $0 \leq t \leq 1$.

We can compute $dx = dt$, $dy = 2t dt$ and $dz = 3t^2 dt$. Thus, the integral can be written as

$$\begin{aligned}\int_C y dx + z dy + x dz &= \int_0^1 [(t^2)(1) + (t^3)(2t) + (t)(3t^2)] dt \\ &= [(t^2)(1) + (t^3)(2t) + (t)(3t^2)]_0^1 = \frac{89}{60}\end{aligned}$$

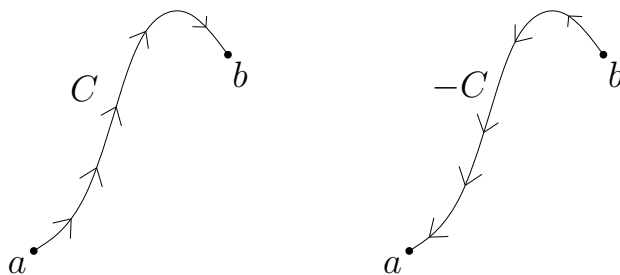


Figure 89: A parametrization of a curve determines an orientation.

18.2.1 Orientation of a parametrization

The parametrization of a curve determines an orientation as shown in Figure 89.

The arc length $ds = |\vec{v}(t)| dt$ is always positive and hence if we take the integral with respect to the arc length, we have

$$\int_C f ds = \int_{-C} f ds$$

But if we take the integral w.r.t. a coordinate variable, then we have:

$$\int_C f dx = - \int_{-C} f dx$$

$$\int_C f dy = - \int_{-C} f dy$$

$$\int_C f dz = - \int_{-C} f dz$$

19 Vector fields

A vector field is a vector function of 3 variables defined as:

$$\vec{F}(x, y, z) = P(x, y, z) \vec{i} + Q(x, y, z) \vec{j} + R(x, y, z) \vec{k}$$

For example, the gradient $\vec{F} = \nabla f$ is a vector field. A fluid velocity $\vec{v}(x, y, z)$ and electromagnetic fields \vec{E} and \vec{B} are examples of vector fields.

Example 76. If $\vec{F}(x, y, z)$ is a force field, compute the work done by \vec{F} .

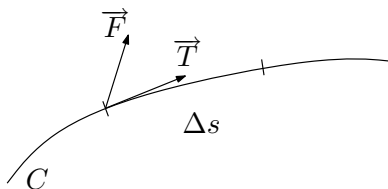


Figure 90: A force field acting on a particle moving along a curve C

The situation is shown in Figure 90. The work done over a small interval Δs is given by:

$$\Delta W = (\vec{F} \cdot \vec{T}) \Delta s$$

where \vec{T} gives the unit tangent vector to the curve C . We can sum the work done over all the intervals and take the limit as $\Delta s \rightarrow 0$ to get

$$W = \int_C (\vec{F} \cdot \vec{T}) ds$$

Alternative formulas: We can find a parametrization to simplify this formula by noting that the unit tangent vector was defined by $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ and that the speed is given by $|\vec{v}| = \frac{ds}{dt}$. Combining these two, we can see that

$$\vec{T} ds = \vec{v} dt$$

Hence, the work done can be written as:

$$W = \int (\vec{F} \cdot \vec{v}) dt$$

If the force field is given by $\vec{F} = (P, Q, R)$, then we have:

$$(\vec{F} \cdot \vec{v}) dt = \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt$$

Further, since $\vec{v} = \frac{d\vec{r}}{dt}$, we can also write this as:

$$W = \int \vec{F} \cdot d\vec{r}$$

Using coordinates, the work can be written as:

$$W = \int \vec{F} \cdot d\vec{r} = \int P dx + Q dy + R dz$$

Don't be fooled by all these formulas! In the end, the same calculations need to be done to calculate the work.

Example 77. Find the work done by:

$$\vec{F} = \frac{k\vec{r}}{|\vec{r}|^3}$$

along the straight segment from $(0, 4, 0)$ to $(0, 4, 3)$.

We parameterize the segment using $x = 0$, $y = 4$, $z = z$. Hence the work done can be computed as

$$W = \int_{z=0}^{z=3} \frac{kz}{r^3} dz = k \int_{z=0}^{z=3} \frac{z}{(16 + z^2)^{3/2}} dz = k \left[\frac{-1}{\sqrt{16 + z^2}} \right]_0^3 = \frac{k}{20}$$

19.1 Application

Let us look at the magnetic field around a wire as shown in Figure 91. Using a line integral, Ampere's law tells us that:

$$\int_C \vec{B} \cdot d\vec{r} = \mu I$$

\vec{B} and $d\vec{r}$ point in the same direction and hence we can write:

$$\vec{B} \cdot d\vec{r} = \vec{B} \cdot \vec{v} dt = |\vec{B}| |\vec{v}| dt$$

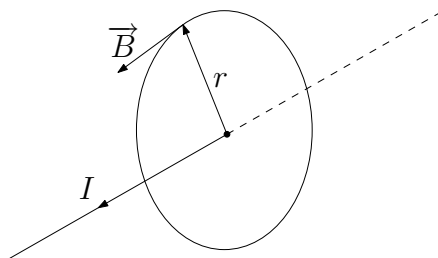


Figure 91: Magnetic field around a wire

We can rewrite the integral as

$$\begin{aligned} \int_C \vec{B} \cdot d\vec{r} &= \int_C |\vec{B}| |\vec{v}| dt \\ &= |\vec{B}| \int_C |\vec{v}| dt \\ &= |\vec{B}| \int_C ds \\ &= 2\pi r |\vec{B}| \end{aligned}$$

So finally:

$$2\pi r |\vec{B}| = \mu I,$$

which gives us

$$|\vec{B}| = \frac{\mu I}{2\pi r}$$

We have proved that the magnetic field decays like $1/r$ away from a wire.

19.2 Fluid flow

If $\vec{F} = (P, Q, 0)$ is a velocity field of a flow, then the **flow integral** for a curve is defined as:

$$\int_C \vec{F} \cdot d\vec{r}$$

If C is a closed curve (a loop), then the flow integral is called the **circulation** and is indicated as:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint P dx + Q dy$$

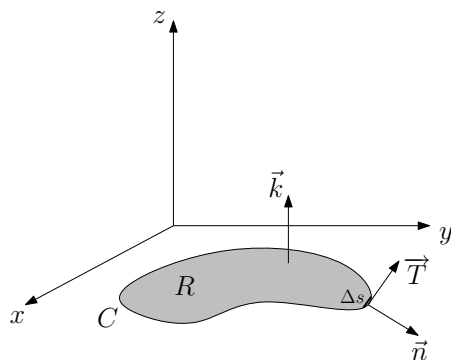
19.3 Flux across a plane curve

Assume we have a plane region R bounded by a smooth closed curve C and a vector field \vec{v} , then we can define the **flux** as the amount of fluid leaving the domain R per unit time. The flux can be expressed mathematically as:

$$\Phi = \oint (\vec{v} \cdot \vec{n}) ds$$

where \vec{n} is a unit vector normal to the curve pointing towards the outside of R . More generally for any vector field \vec{F} we define the flux as:

$$\Phi = \oint (\vec{F} \cdot \vec{n}) ds$$

Figure 92: Defining flux across a plane region R

An intuitive explanation of this definition can be derived by considering a small element of the region R as shown in Figure 92. To compute the flux, we look at this small element of length Δs at the boundary, as shown closely in Figure 93.

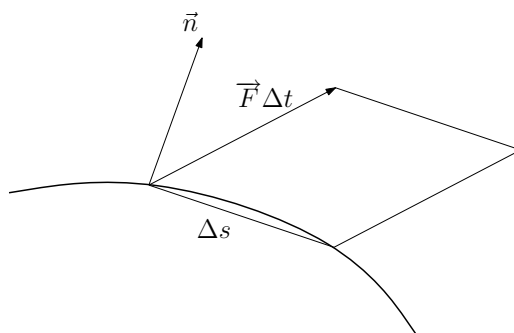


Figure 93: The flux through a small region of the boundary.

The area of the parallelogram in Figure 93 gives the amount of fluid leaving/entering the small region of the boundary Δs in time Δt . The area of the parallelogram is given by:

$$\text{Area} = \text{Base} \times \text{Height} = \Delta s \left((\vec{F} \Delta t) \cdot \vec{n} \right)$$

Hence, the net flow per unit time over the small region of the boundary is given by $(\vec{F} \cdot \vec{n}) \Delta s$. Summing up the flow over the entire boundary, we have:

$$\text{Net flow per unit time} = \Phi = \oint_C (\vec{F} \cdot \vec{n}) ds$$

We can see from Figure 92 that $\vec{T} \times \vec{k} = \vec{n}$. Also we have $\vec{T} = \frac{1}{|\vec{v}|}(x', y', z')$ and consequently

$$\vec{n} = \frac{1}{|\vec{v}|}(x', y', z') \times (0, 0, 1) = \frac{1}{|\vec{v}|}(y', -x', 0)$$

This implies that we can write

$$\vec{F} \cdot \vec{n} = (P, Q, 0) \cdot \left[\frac{1}{|\vec{v}|}(y', -x', 0) \right] = \frac{1}{|\vec{v}|} (Py' - Qx')$$

where $\vec{F} = (P, Q)$. Now, we can compute the flux as

$$\begin{aligned}\Phi &= \oint (\vec{F} \cdot \vec{n}) ds \\ &= \oint \frac{1}{|\vec{v}|} \left(P \frac{dy}{dt} - Q \frac{dx}{dt} \right) ds \\ &= \oint \left(P \frac{dy}{dt} - Q \frac{dx}{dt} \right) dt \\ &= \oint P dy - Q dx\end{aligned}$$

Recall that we had defined the circulation as:

$$\oint (\vec{F} \cdot \vec{T}) ds = \int P dx + Q dy$$

Comparing these equations, we can see that the flux of (P, Q) is the same as the circulation of $(-Q, P)$. Note that, in fact, (P, Q) maps onto $(-Q, P)$ by a rotation of $\pi/2$ while \vec{n} maps onto \vec{T} . This shows that these two definitions are actually closely related by a simple rotation. We will come back to this point when discussing the Green's theorem.

This remark, however, does not extend to the 3D case.

Example 78. Find the flux of $\vec{F} = (x - y)\vec{i} + x\vec{j}$ across the unit circle.

We can parameterize the unit circle using: $x = \cos t$ and $y = \sin t$. This gives us $dx = -\sin t dt$ and $dy = \cos t dt$. Now, we can compute the flux as

$$\begin{aligned}\Phi &= \int -Q dx + P dy \\ &= \int_0^{2\pi} ((\cos t - \sin t) \cos t - \cos t(-\sin t)) dt \\ &= \int_0^{2\pi} \cos^2 t dt = \pi\end{aligned}$$

Reading: Thomas - 16.3, 16.4

20 Potential functions and conservative fields

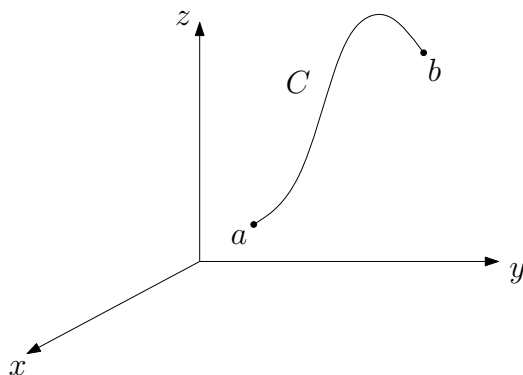
Recall that in 1D, we have the fundamental theorem of calculus:

$$\int_a^b f'(x) dx = f(b) - f(a)$$

A similar theorem can be derived for vector fields, which says that if we have a vector field \vec{F} defined on a region D such that \vec{F} is of the form $\vec{F} = \nabla f$ for some function f , then we have:

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = f(b) - f(a)}$$

where a denotes the starting point of the path C and b denotes the ending point of C , as shown in Figure 94. Such a vector field \vec{F} is called **conservative** and the function f is said to be the **potential function** of \vec{F} .

Figure 94: A given path C starting at a and ending at b

We can derive the above relation by first noting that for a function $f(x, y, z)$, we have:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

Now, we can use the definition of the line integral to write

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (\nabla f \cdot \vec{v}) dt \\ &= \int_0^1 \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \\ &= \int_0^1 \frac{df}{dt} dt \\ &= f(b) - f(a) \end{aligned}$$

where a is the point at $t = 0$ and b is the point at $t = 1$.

Example 79. Find the work done by the conservative force

$$\vec{F} = -\frac{mMG}{|\vec{r}|^3} \vec{r}$$

to go from $(3, 4, 12)$ to $(2, 2, 1)$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

We can check that the potential function for the conservative force field \vec{F} is given by $f = \frac{mMG}{|\vec{r}|}$ as $\vec{F} = \nabla f$. Thus, the work done is given by:

$$\begin{aligned} F &= \int_C \vec{F} \cdot d\vec{r} \\ &= f(2, 2, 1) - f(3, 4, 12) \\ &= mMG \left(\frac{1}{\sqrt{4 + 4 + 1}} - \frac{1}{\sqrt{9 + 16 + 144}} \right) \\ &= mMG \left(\frac{1}{3} - \frac{1}{13} \right) \\ &= \frac{10mMG}{39} \end{aligned}$$

20.1 Work done by conservative fields

Note that as a consequence of the theorem, we can see that the line integral of a conservative field \vec{F} is **independent** of the path as the value of the integral depends only on the value of the potential function f at the end points. Hence, the value of the integral along *any* path from point a to b is the same:

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = f(b) - f(a)$$

Also, if \vec{F} is conservative, then the value of the integral along any **closed** path C is 0:

$$\oint_C \vec{F} \cdot d\vec{r} = f(a) - f(a) = 0$$

Example 80. Find the work done by the vector field $\vec{w} = -y\vec{i} + x\vec{j}$ over the closed semi-circular path C shown in Figure 95.

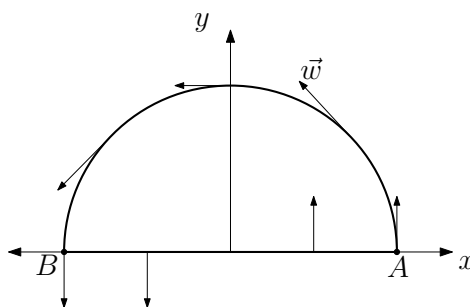


Figure 95: A vector field \vec{w} along a semi-circular path C (the thick line)

We will divide the path in two parts: C_1 and C_2 , C_1 going from A to B over the circular path and C_2 coming back from B to A over the straight path. Note that the work done is given by:

$$\int_C \vec{w} \cdot d\vec{r} = \int_C (-y) dx + x dy = \int_{C_1} (-y) dx + x dy + \int_{C_2} (-y) dx + x dy$$

For the path C_1 , parameterize the curve by $x = \cos \theta$ and $y = \sin \theta$ for $\theta : 0 \rightarrow \pi$. Thus, we have $-y dx = \sin^2 \theta d\theta$ and $x dy = \cos^2 \theta d\theta$. This gives us the work done for C_1 as:

$$\int_{C_1} (-y) dx + x dy = \int_0^\pi (\sin^2 \theta + \cos^2 \theta) d\theta = \pi$$

For the path C_2 , note that $y = 0$ and also that $dy = 0$ as the y coordinate is constant. Hence, the work done over C_2 is given by:

$$\int_{C_2} (-y) dx + x dy = 0$$

Thus, the total work done by the field over the closed path is:

$$\int_C \vec{w} \cdot d\vec{r} = \pi$$

But this is nonzero even though we went over a **closed** path and hence we can conclude that the vector field \vec{w} is not conservative.

Example 81. Find the work done by $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$ from $(0,0,1)$ to $(1,1,1)$.

1. We parameterize the curve by: $x = x$, $y = x$, $z = 1$. We can write the work done as

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C yz dx + xz dy + xy dz$$

Now, for the given curve, we compute $dx = dx$, $dy = dx$ and $dz = 0$. Thus, we can write the work done as:

$$W = \int_0^1 (x)(1) dx + x(1) dx + 0 = [x^2]_0^1 = 1$$

2. We parameterize the curve by: $x = x$, $y = x^2$, $z = 1$. As before, for the given curve, we compute $dy = 2x dx$ and $dz = 0$. Thus, we can write the work done as:

$$W = \int_0^1 (x^2)(1) dx + x(1)(2x) dx + 0 = 3 \int_0^1 x^2 = 1$$

3. We now use the scalar potential. First, we must find it: we want a function f such that $\vec{F} = \nabla f$. This means that:

$$\frac{\partial f}{\partial x} = yz, \quad \frac{\partial f}{\partial y} = xz, \quad \frac{\partial f}{\partial z} = xy$$

If we integrate the first equation w.r.t. x , then we have:

$$f = xyz + g(y, z)$$

which can now be used in the second equation to get

$$\frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz$$

which gives us $\frac{\partial g}{\partial y} = 0$ and hence $g(y, z)$ is only a function of z . We denote it by $h(z)$. Finally, using the third equation, we find:

$$\frac{\partial f}{\partial z} = xy + h'(z) = xy$$

which gives us $h' = 0$. h is therefore a constant which can set to 0. Then, our scalar potential is $f = xyz$. Thus, we can compute the work done as

$$W = f(1, 1, 1) - f(0, 0, 1) = 1 - 0 = 1$$

This is the same result as before!

20.2 Simply connected regions

A **connected** region implies a region which is in one piece, i.e. there is a path from every point to every other point which lies completely within the region. A **simply connected** region has the additional condition that there are no holes in the region. For example, in Figure 96, regions 2 and 3 are not simply connected as they have holes while region 1 is simply connected.

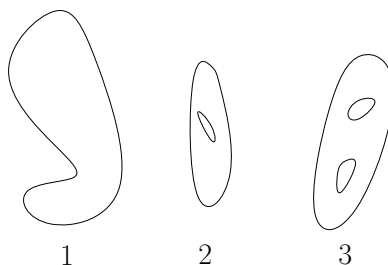


Figure 96: Three regions, two of which have holes

20.3 Checking for conservative fields

If we want to ascertain that a vector field is conservative, we can use the following theorem to do this: a vector field $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ defined over a simply connected region is conservative if and only if the following equations hold:

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}}$$

To get a sense of why these equations must hold if \vec{F} is conservative, consider a potential function f such that $\vec{F} = \nabla f$. Thus, we have $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$. Now, differentiating M w.r.t. y and N w.r.t. x , we get $\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$ respectively. But we know that the order of the derivatives can be switched and hence we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Similarly, we can prove the other equations are satisfied.

Showing the reverse is more complicated and will be omitted.

Example 82. Show that the vector field

$$\vec{F} = (e^x \cos y + yz)\vec{i} + (xz - e^x \sin y)\vec{j} + (xy + z)\vec{k}$$

is conservative and find its scalar potential.

We will use the theorem to test if \vec{F} is conservative. Note that here: $M = e^x \cos y + yz$, $N = xz - e^x \sin y$ and $P = xy + z$. We can calculate:

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y}$$

Hence, \vec{F} is conservative.

To find the scalar potential for \vec{F} , we can integrate the following equations:

$$\frac{\partial f}{\partial x} = e^x \cos y + yz \tag{11}$$

$$\frac{\partial f}{\partial y} = xz - e^x \sin y \tag{12}$$

$$\frac{\partial f}{\partial z} = xy + z \tag{13}$$

Integrating equation 11, we get:

$$f = e^x \cos y + xyz + g(y, z)$$

where g is a (yet) unknown function of y and z . To find g , we use the other two equations above in the following manner. First, we compute $\frac{\partial f}{\partial y}$ using this equation as:

$$\frac{\partial f}{\partial y} = -e^x \sin y + xz + \frac{\partial g}{\partial y}$$

Equating this with equation 12, we get

$$\frac{\partial g}{\partial y} = 0$$

which means that g is only a function of z , denoted by $h(z)$. Thus, we can now write our function as

$$f = e^x \cos y + xyz + h(z)$$

We can now repeat the same trick, computing $\frac{\partial f}{\partial z}$ using this equation as:

$$\frac{\partial f}{\partial z} = xy + h'(z)$$

Equating this with equation 13, we get

$$h'(z) = z$$

which can now be integrated to get

$$h(z) = \frac{z^2}{2} + C$$

where C is an unknown constant. We can set it to 0. Thus, we found our scalar potential as:

$$f = e^x \cos y + xyz + \frac{z^2}{2}$$

20.4 Application: Conservation of Energy

Recall that we can express a conservative force field \vec{F} in terms of the potential energy U as $\vec{F} = -\nabla U$. Suppose we want to compute the work done by the force \vec{F} from point a to b . Since \vec{F} is conservative, we can write the work done as

$$W = U(a) - U(b)$$

From the definition of work, we can also write the work done as

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \vec{v}) dt$$

Newton's third law gives us $\vec{F} = m \frac{d\vec{v}}{dt}$, and using this, we can write:

$$W = m \int_C \left(\frac{d\vec{v}}{dt} \cdot \vec{v} \right) dt = \frac{m}{2} \int_C \frac{d}{dt} (|\vec{v}|^2) dt = \frac{m}{2} (|\vec{v}(b)|^2 - |\vec{v}(a)|^2)$$

where the last equality is obtained using the fundamental theorem of calculus.

Now, we can equate the two expressions for work and rearrange terms to get:

$$\frac{m}{2}|\vec{v}(b)|^2 + U(b) = \frac{m}{2}|\vec{v}(a)|^2 + U(a)$$

Note that this equation is valid for any two points a and b in the domain. Hence, this can be rephrased as

$$\frac{m}{2}|\vec{v}|^2 + U = \text{constant}$$

which is the law of conservation of energy! This explains the use of the adjective *conservative* for the force field.

21 Green's theorem

Assume that we are given a simple, closed, plane curve C which encloses a region R as shown in Figure 97. Here *simple* means that the curve has no self-intersections. We will also assume that the curve has **positive** orientation, i.e. the region R lies to the **left** of the tangent vector \vec{T} at every point, as shown in Figure 97.

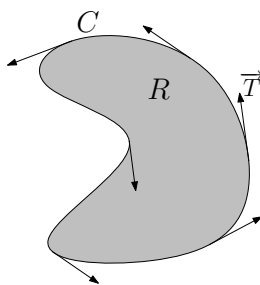


Figure 97: A simple, closed, plane curve C with positive orientation

Green's theorem gives us a way to convert a line integral along the curve C to a double integral over the region R .

21.1 Circulation-curl form

The *circulation-curl* form of **Green's theorem** states that given two functions M and N that are smooth and differentiable over the region R enclosed by a simple, closed, planar curve C , we can write

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Proof. We can prove Green's theorem by showing that

$$\oint_C M dx = - \iint_R \frac{\partial M}{\partial y} dx dy \quad \text{and} \quad \oint_C N dy = \iint_R \frac{\partial N}{\partial x} dx dy$$

We will only prove the first equation as the proof for the other one is very similar. For proving this, we divide the curve C into 2 parts: C_1 and C_2 , where C_1 goes from $x = a$ to $x = b$ and C_2 goes from $x = b$ to $x = a$ as shown in Figure 98.

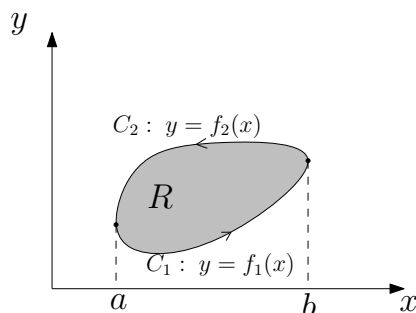


Figure 98: Dividing a simple, closed, plane curve C into two parts C_1 and C_2

Using the fundamental theorem of calculus, we can compute the integral over the region R as follows:

$$\begin{aligned}
 - \iint_R \frac{\partial M}{\partial y} dx dy &= - \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy dx \\
 &= - \int_a^b (M(x, f_2(x)) - M(x, f_1(x))) dx \\
 &= \int_a^b M(x, f_1(x)) dx + \int_b^a M(x, f_2(x)) dx \\
 &= \int_{C_1} M dx + \int_{C_2} M dx \\
 &= \oint_C M dx
 \end{aligned}$$

which is what we wanted to prove and hence we are done! \square

21.2 Flux-divergence form

The *flux-divergence* form of **Green's theorem** states that given two functions M and N that are smooth and differentiable over the region R enclosed by a simple, closed, planar curve C , we can write

$$\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy$$

Proof. We can prove this form using the circulation-curl form. Choose two functions $P = -N$ and $Q = M$. The circulation-curl form holds and we have:

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

In terms of M and N , this gives us:

$$\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy$$

This is what we set out to prove! \square

Example 83. Recall example 80. Find the work done by the vector field $\vec{w} = -y\vec{i} + x\vec{j}$ over the closed semi-circular path C shown in Figure 95.

We will use Green's theorem to compute the line integral for work:

$$W = \oint_C \vec{F} \cdot d\vec{r}$$

If we write $M = -y$ and $N = x$, then we have

$$W = \oint_C M dx + N dy$$

Using the circulation-curl form of Green's theorem, we have

$$W = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R 2 dx dy = 2 \cdot \text{Area}$$

But the area of the given semi-circular region R is $\pi/2$ and hence:

$$W = \pi$$

which is the same as what we obtained using explicit integration in example 80.

Example 84. Compute the integral

$$I = \oint_C \left(2y + \sqrt{9 + x^3} \right) dx + \left(5x + e^{\arctan y} \right) dy$$

where C is a circle of radius 2.

We set $M = 2y + \sqrt{9 + x^3}$ and $N = 5x + e^{\arctan y}$. We use the circulation-curl form of Green's theorem to write

$$I = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = 3 \cdot \text{Area}$$

The area bounded by the curve C is $\pi(2)^2 = 4\pi$ and hence $I = 12\pi$.

Using the Green's theorem can lead to simple integrals and easy answers.

21.3 Area bounded by a simple, closed, plane curve

Up until now we have used Green's theorem to convert a line integral into a double integral, but sometimes we might want to go the other way, when computing the line integral is simpler than the double integral. Suppose we want to compute the area of a region R which can be expressed as

$$A = \iint_R dx dy$$

Suppose that we can find functions M and N such that

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$$

then using Green's theorem, we can write the area as

$$A = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \oint_C M dx + N dy$$

One simple choice of M and N that satisfies this criterion is $M = 0$ and $N = x$. Another possible choice is $M = -y$ and $N = 0$. Using these choices along with Green's theorem, we can write the area as:

$$A = \oint_C x dy = - \oint_C y dx \tag{14}$$

Example 85. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Firstly, we want to parameterize the ellipse such that the area lies to the left of the parameterized curve. Such a parametrization is given by $x = a \cos \theta$ and $y = b \sin \theta$. This gives us $dy = b \cos \theta d\theta$. Now, using Equation (14), we can write the area as:

$$A = \oint_C x dy = \int_0^{2\pi} ab \cos^2 \theta d\theta = \pi ab$$

Example 86. Compute

$$\oint_C xy dy - y^2 dx$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

We will use the flux-divergence form of Green's theorem by choosing $M = xy$ and $N = y^2$. Then the given integral can be computed as follows:

$$\begin{aligned} \oint_C M dy - N dx &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \\ &= \int_0^1 \int_0^1 (y + 2y) dx dy = \int_0^1 [3xy]_0^1 dy \\ &= \int_0^1 3y dy = \left[\frac{3y^2}{2} \right]_0^1 = \frac{3}{2} \end{aligned}$$

Reading: Thomas - 16.4, 16.5, 16.6, 16.7, 16.8

21.4 Piston engine

We want to find an equation of the work done by a piston engine. A piston engine refers to an engine in which a piston compresses gas in a chamber as shown in figure 99(a). A piston engine has a P - V diagram as shown in figure 99(b).

If the cross-sectional area of the piston is A , then the work done by the piston as it moves by a small height Δz is given by

$$\Delta W = F \Delta z$$

where F is the force exerted by the piston. But the force is also given by $F = PA$, where P is the pressure of the gas inside the chamber. Hence, we can write

$$\Delta W = PA \Delta z$$

Now, note that the change in volume is given by $\Delta V = A\Delta z$, and hence we have $\Delta W = P\Delta V$. Summing the work done over a change in height, we get:

$$W = \int P dV$$

For the complete piston engine cycle, we have the P - V diagram shown in figure 99(b) and the work done is given by the area under the curve. Recall that we derived the area enclosed by a curve as

$$\iint_R dx dy = \oint_C x dy$$

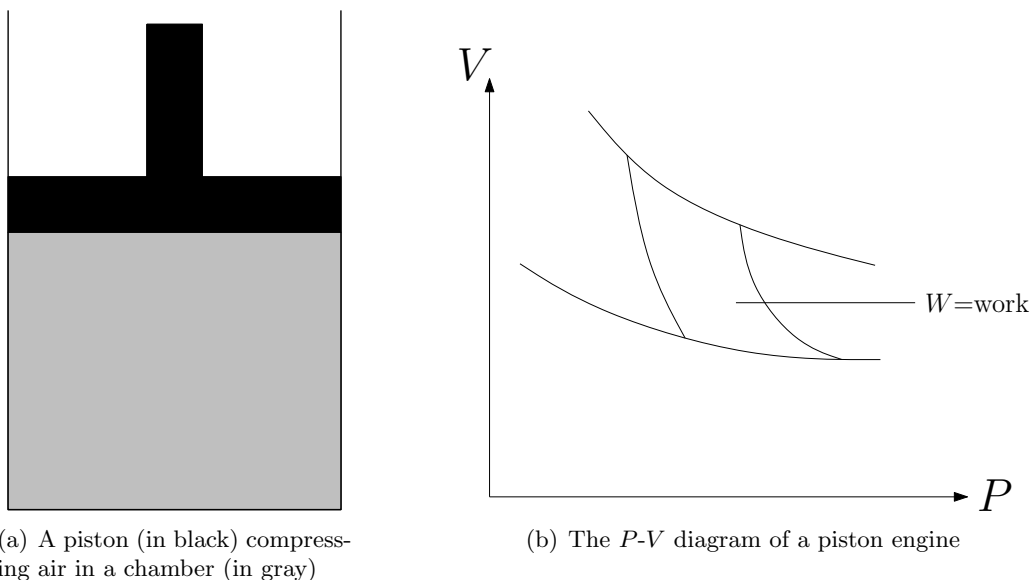


Figure 99: The working of a piston engine

and here, the coordinates x and y are P and V respectively, giving us the work done as the area enclosed by the curve:

$$W = \oint_C P dV = \iint dP dV$$

which is the same expression as what we had previously used to study the Carnot cycle!

21.5 Incompressible flow

The flux Φ per unit time of a vector field \vec{F} over a closed plane region R is defined as:

$$\text{Net flow per unit time} = \Phi = \oint (\vec{F} \cdot \vec{n}) ds$$

For a fluid flow with velocity field given by \vec{v} , the flux is equal to:

$$\text{Net flow} = \oint (\vec{v} \cdot \vec{n}) ds \quad (15)$$

We say that a flow is *incompressible* if the density of the fluid does not change over time. Another way to say this is that the fluid particles do not accumulate inside the region and consequently the amount of fluid entering the region must be equal to the amount of fluid that is going out. Referring to figure 100, the incompressibility condition says that

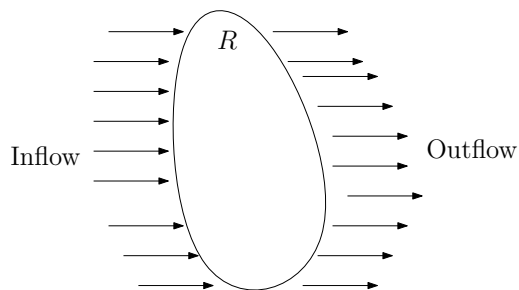
$$\text{Inflow} = \text{Outflow}$$

which in turn implies that

$$\text{Net Flow} = 0$$

Combining the incompressibility condition with equation 15 we have that

$$\text{Net Flow} = \oint (\vec{v} \cdot \vec{n}) ds = 0$$

Figure 100: A flow across a region R

Consequently, if the velocity is given by $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$, we get:

$$\oint_C v_1 dy - v_2 dx = 0$$

Using the flux-divergence form of Green's theorem, we can write this as a double integral:

$$\iint_R \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) dx dy = 0$$

Note that this equation must hold for *any* simple region R and hence we must have the following condition to hold true for incompressible flow:

$$\boxed{\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0}$$

21.6 Generalization of Green's theorem

We can extend Green's theorem to hold for regions with holes (i.e., regions that are not simply connected) if we choose positive orientation for the outer boundary and negative orientation for all the inner curves, as shown in figure 101.

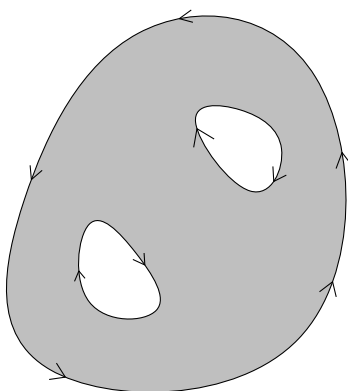
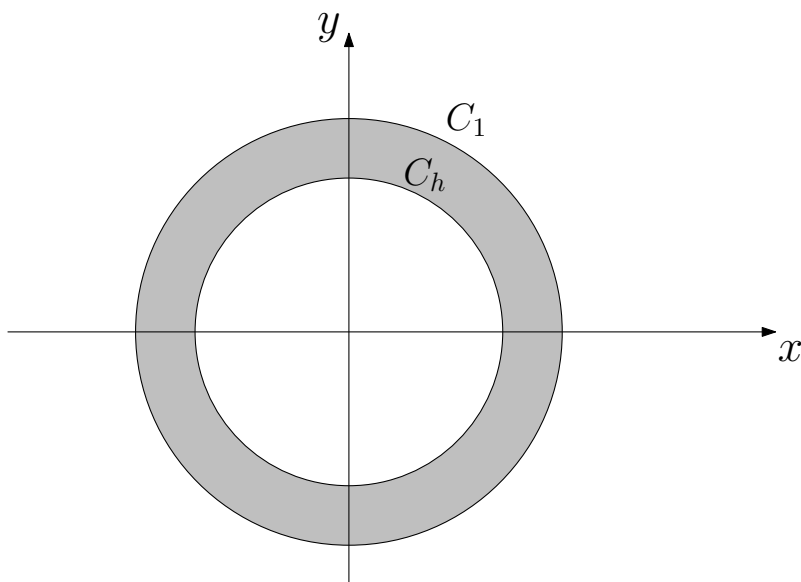


Figure 101: Extending Green's theorem for a region (shaded) with holes

Example 87. Verify the circulation form of Green's theorem on the annular ring: $h^2 \leq x^2 + y^2 \leq 1$, $0 < h < 1$, if:

$$\vec{F} = -\frac{y}{x^2 + y^2} \vec{i} + \frac{x}{x^2 + y^2} \vec{j}$$

Figure 102: The annular region R (shaded)

The annular region has an outer boundary $C_1 : x^2 + y^2 = 1$ and an inner boundary $C_2 : x^2 + y^2 = h^2$ as shown in figure 102. We put $M = -\frac{y}{x^2+y^2}$ and $N = \frac{x}{x^2+y^2}$. We have to verify the circulation-curl form of Green's theorem, which says that:

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

We will evaluate both sides independently given the functions M and N , and the curve $C = C_1 \cup C_2$. First, we can evaluate the right hand side to be:

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R \left(\frac{y^2 - x^2}{x^2 + y^2} - \frac{y^2 - x^2}{x^2 + y^2} \right) dx dy = 0$$

To evaluate the line integral on the left hand side of Green's theorem, we parameterize the two curves keeping in mind that the outer curve should have positive orientation and the inner curve should have negative orientation: For C_1 , we put $x = \cos t$ and $y = \sin t$ where $0 \leq t \leq 2\pi$. For C_2 , we put $x = h \cos \theta$ and $y = -h \sin \theta$ where $0 \leq \theta \leq 2\pi$. Thus, the line integral becomes:

$$\begin{aligned} \oint_C M dx + N dy &= \oint_{C_1} \frac{x dy - y dx}{x^2 + y^2} + \oint_{C_2} \frac{x dy - y dx}{x^2 + y^2} \\ &= \int_0^{2\pi} (\cos^2 t + \sin^2 t) dt - \int_0^{2\pi} \frac{h^2 (\cos^2 \theta + \sin^2 \theta)}{h^2} d\theta \\ &= 2\pi - 2\pi \\ &= 0 \end{aligned}$$

We have verified the circulation-curl form of Green's theorem for this case. Note that the functions M and N are not defined at $(0,0)$ and hence we need the hole to be present in the region in order to apply Green's theorem!

Remark: Let's see what would happen if the outer boundary C_1 is an arbitrary curve instead of being a circle, as shown in figure 103. Since we have the same M and N , using the results above,

we can write Green's theorem as:

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = 0$$

The integral on the left can be split up as we did above to get:

$$\oint_C M dx + N dy = \oint_{C_1} M dx + N dy + \oint_{C_h} M dx + N dy = 0$$

But we evaluated

$$\oint_{C_h} M dx + N dy = -2\pi$$

independent of C_1 . Hence, we get

$$\oint_{C_1} M dx + N dy = 2\pi$$

for all the curves C_1 that surround C_h ! This can also be derived directly if we parameterize C_1 as $x = r \cos \theta$ and $y = r \sin \theta$, where both r and θ vary along the curve. Then, we get $dx = -r \sin \theta d\theta + \cos \theta dr$ and $dy = r \cos \theta d\theta + \sin \theta dr$. Now we can compute our integral as

$$\oint_{C_1} M dx + N dy = \oint_{C_1} \frac{x dy - y dx}{x^2 + y^2} = \oint_{C_1} \frac{r^2(\cos^2 \theta + \sin^2 \theta)}{r^2} d\theta = 2\pi$$

We get the same thing!

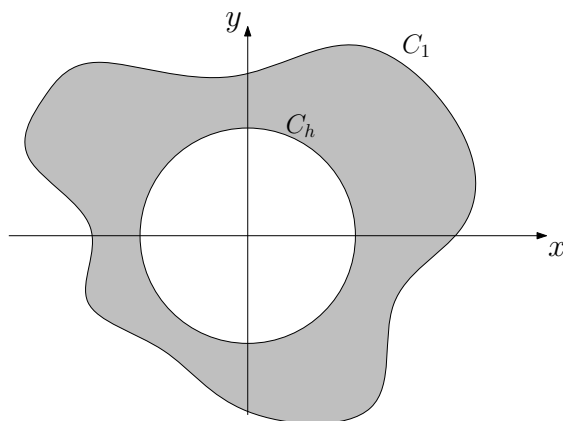


Figure 103: A region R (shaded) bounded by an arbitrary curve C_1

Example 88. Find the area bounded by the folium of Descartes in the first quadrant (shown in figure). The equation of the curve is: $x^3 + y^3 = 3xy$.

We will first try to parameterize the curve. Suppose we have $y = tx$. This gives us:

$$x = \frac{3t}{1+t^3}, \quad y = \frac{3t^2}{1+t^3}$$

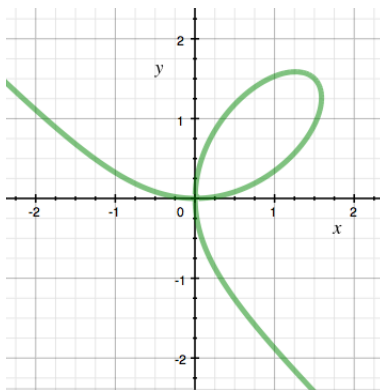


Figure 104: The folium of Descartes (source: Wikipedia)

The curve in the first quadrant corresponds to t going from 0 to ∞ (you should convince yourself about this). We can also compute $dy = \left(\frac{6t}{1+t^3} - \frac{9t^4}{(1+t^3)^2} \right) dt$. Invoking the formula for computing the area that we derived using Green's theorem, we get:

$$\begin{aligned}
 \text{Area} &= \oint_C x \, dy = \int_0^\infty \left(\frac{3t}{1+t^3} \right) \left(\frac{6t}{1+t^3} - \frac{9t^4}{(1+t^3)^2} \right) dt \\
 &= \int_0^\infty \frac{18t^2}{(1+t^3)^2} dt - \int_0^\infty \frac{27t^5}{(1+t^3)^3} dt \\
 &= \int_1^\infty \frac{9}{u^2} du - \int_1^\infty \frac{9(u-1)}{u^3} du \quad (\text{set } u = 1+t^3) \\
 &= 9 \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{3}{2}
 \end{aligned}$$

22 Surface integrals

We have seen the line integrals earlier which integrate a function along a line. Surface integrals are an extension of line integrals that allow us to integrate a function over a surface. Mathematically, if we have a surface S as shown in figure 105 then the integral is given by:

$$\iint_S f(x, y, z) \, dS$$

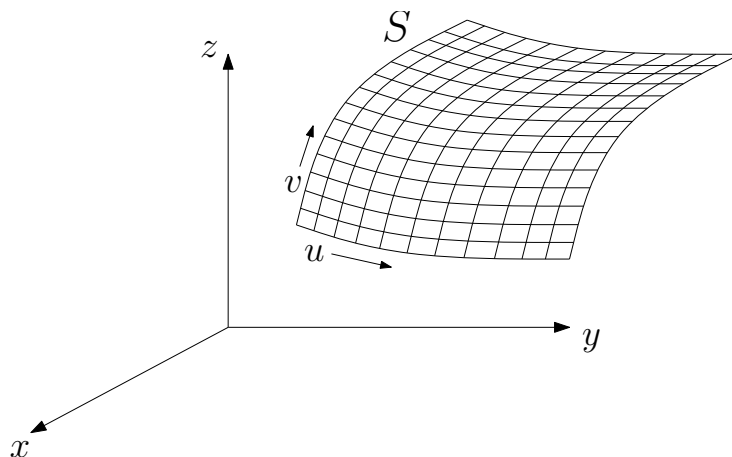
22.1 General formula

The most generic representation for a surface is given by a parameterized form of a surface: $x(u, v)$, $y(u, v)$ and $z(u, v)$, where u and v increase along the edges of the surface as shown in figure 105.

Example 89. If we have a cylinder of radius a , then it can be parameterized using two parameters θ and z as: $x = a \cos \theta$, $y = a \sin \theta$, $z = z$.

Example 90. A cone $z = \sqrt{x^2 + y^2}$ can be parameterized using two parameters θ and r as: $x = r \cos \theta$, $y = r \sin \theta$ and $z = r$.

Example 91. A sphere $x^2 + y^2 + z^2 = a^2$ can be parameterized using two parameters θ and ϕ as: $x = a \sin \phi \cos \theta$, $y = a \sin \phi \sin \theta$ and $z = a \cos \phi$.

Figure 105: A generic surface S

22.2 Definition of surface integrals

The surface S can be divided into small regions as shown in figure 105. As before, we can write the surface integral as a Riemann sum by taking the limit as both Δu and Δv approach zero and summing over all the elements:

$$\iint_S f(x, y, z) dS = \lim_{\substack{\Delta u \rightarrow 0 \\ \Delta v \rightarrow 0}} \sum_i f_i \Delta A_i$$

To compute the area of the small region, we consider one such small region and approximate it by a parallelogram as shown in figure 106. The area of the parallelogram in figure 106 is given by:

$$\Delta A = \left| \left(\Delta u \frac{\partial \vec{r}}{\partial u} \right) \times \left(\Delta v \frac{\partial \vec{r}}{\partial v} \right) \right| = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta u \Delta v$$

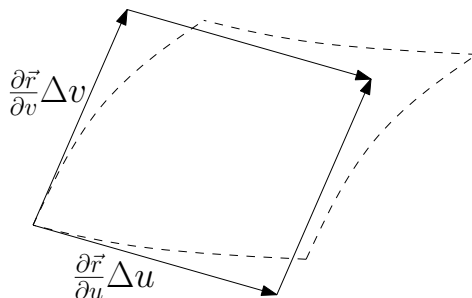


Figure 106: Approximating a small region (dashed) with a parallelogram. The difference shown is exaggerated for illustration.

Plugging this value of the area in the expression for the surface integral, we get:

$$\boxed{\iint_S f(x, y, z) dS = \iint_R f(x, y, z) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv}$$

Example 92. Find the surface area and the moment of inertia of a hemisphere of radius R about the z -axis.

We can parameterize the surface of a hemisphere as: $x = R \sin \phi \cos \theta$, $y = R \sin \phi \sin \theta$ and $z = R \cos \phi$, where $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$. Since we have $u = \phi$ and $v = \theta$ here, we can compute

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = R^2 \sin \phi$$

Consequently, we can compute the area as:

$$\text{Area} = \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=2\pi} R^2 \sin \phi \, d\theta \, d\phi = 2\pi R^2 [-\cos \phi]_0^{\pi/2} = 2\pi R^2$$

We can also find the moment of inertia as follows (assuming that density $\delta = 1$):

$$\begin{aligned} I_z &= \iint (x^2 + y^2) \delta \, dS \\ &= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=2\pi} (R^2 \sin^2 \phi)(R^2 \sin \phi) \, d\theta \, d\phi \\ &= 2\pi R^4 \int_0^{\pi/2} \sin^3 \phi \, d\phi = \frac{4\pi}{3} R^4 \end{aligned}$$

We can also see that the mass of the hemisphere of unit density is $M = 2\pi R^2$ and hence we can write the moment of inertia as $I_z = \frac{2M}{3} R^2$. Thus, the radius of gyration is given as:

$$r_z = \sqrt{\frac{I_z}{M}} = R\sqrt{\frac{2}{3}}$$

22.3 Special cases

22.3.1 Case 1

Let's say we are given a surface $z = f(x, y)$ and we want to integrate a function $g(x, y, z)$ over the surface. Note that the parametrization is given here as: $x = x$, $y = y$ and $z = f(x, y)$. Thus, we have $\vec{r}(x, y) = (x, y, f(x, y))$. Now we can compute:

$$\frac{\partial \vec{r}}{\partial x} = \left(1, 0, \frac{\partial f}{\partial x} \right)$$

$$\frac{\partial \vec{r}}{\partial y} = \left(0, 1, \frac{\partial f}{\partial y} \right)$$

This gives us the cross product:

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$$

Thus we can write the surface integral as:

$$\boxed{\iint_S f(x, y, z) \, dS = \iint_R g(x, y, z) \left(\sqrt{1 + \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2} \right) dx \, dy}$$

22.3.2 Case 2

If we are given an implicit surface $F(x, y, z) = 0$, and we assume that z can be written in terms of x and y as $z = f(x, y)$, then we can write

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

and similarly

$$\frac{\partial f}{\partial y} = -\frac{F_y}{F_z}$$

Hence, we have

$$\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1 + \left(\frac{F_x}{F_z}\right)^2 + \left(\frac{F_y}{F_z}\right)^2} = \frac{|\nabla F|}{|\nabla F \cdot \vec{k}|}$$

Thus, we can write the surface integral as:

$$\boxed{\iint_S f(x, y, z) dS = \iint_R g(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \vec{k}|} dx dy}$$

Example 93. Find the centroid of the upper hemisphere of radius R .

From symmetry it is clear that $\bar{x} = \bar{y} = 0$ and $0 < \bar{z} < R$. We know that the mass of the hemisphere is given by $M = 2\pi R^2$. Now, we can compute:

$$\bar{z} = \frac{1}{M} \iint z dS$$

We know that for the hemisphere, $z = \sqrt{R^2 - x^2 - y^2}$. We can apply case 1:

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

Now, we can compute

$$1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{x^2 + y^2 + z^2}{z^2} = \left(\frac{R}{z}\right)^2$$

Thus, we have:

$$\bar{z} = \frac{1}{M} \iint z \frac{R}{z} dx dy = \frac{R}{2\pi R^2} \iint dx dy$$

where the integral is over a disc of radius R and hence the value of the integral is πR^2 , resulting in

$$\bar{z} = \frac{R}{2\pi R^2} \pi R^2 = \frac{R}{2}$$

23 Flux of a vector field

We will define the flux of a vector field over surfaces, which is analogous to the way we previously defined flux across planar curves. There are additional requirements for being able to make this extension though, as there are two possible kinds of surfaces:

1. One-sided surface: The normal vector \vec{n} cannot be defined for this kind of a surface. An example of such a surface is the möbius strip as shown on the top in figure 107.
2. Two-sided surface: It is possible to define a vector \vec{n} on the surface such that it is normal to the surface and continuous. This is the bottom surface in figure 107.

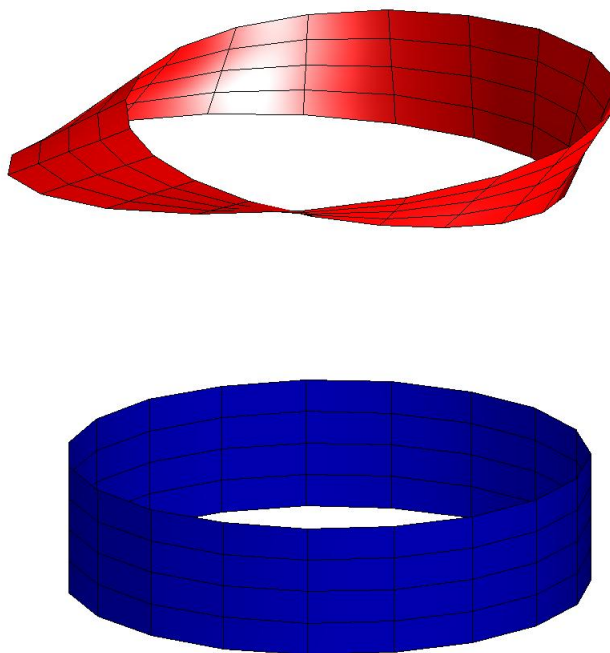


Figure 107: The möbius strip shown on top with a cylinder shown on the bottom.

We will restrict our attention only to two sided or orientable surfaces as we will need to work with the surface normal. Let's say we want to define a flux integral of a vector field \vec{F} over a two sided surface S with the surface normal \vec{n} as:

$$\Phi = \iint_S \vec{F} \cdot \vec{n} \, dS$$

As before, we can look at a small region of the surface, approximate it with a parallelogram and observe that we can compute the normal vector as:

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}$$

Thus, we can write the flux as:

$$\Phi = \iint_S \vec{F} \cdot \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|} \, du \, dv$$

This is equal to:

$$\Phi = \iint_S \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du dv$$

23.1 Special cases

We now assume that the force is given by $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$.

23.1.1 Case 1

If the surface is expressible in the form $z = f(x, y)$ then we can write the flux as:

$$\Phi = \iint_S \left(-M \frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} + P \right) dx dy$$

23.1.2 Case 2

If the surface is given as $G(x, y, z) = 0$ then we can write the flux as:

$$\Phi = \iint_S \frac{M \frac{\partial G}{\partial x} + N \frac{\partial G}{\partial y} + P \frac{\partial G}{\partial z}}{\left| \frac{\partial G}{\partial z} \right|} dx dy$$

Remark: Make sure that the normal vector that you use points in the right direction. If $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is pointing in the direction opposite to \vec{n} , then change its sign.

Example 94. Find the flux $\iint_S \vec{F} \cdot \vec{n} dS$ of $\vec{F} = z\vec{k}$ across the surface of a hemisphere of radius R .

Choose the parametrization $x = R \sin \phi \cos \theta$, $y = R \sin \phi \sin \theta$ and $z = R \cos \phi$. We have

$$\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = R^2 \sin \phi \vec{n} = R \sin \phi \vec{r}$$

where $\vec{n} = \frac{\vec{r}}{R}$. Thus, we get

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} dS &= \iint \vec{F} \cdot (R \sin \phi \vec{r}) d\theta d\phi = \iint (z\vec{k}) \cdot (R \sin \phi \vec{r}) d\theta d\phi \\ &= \iint z^2 R \sin \phi d\theta d\phi = \iint R^3 \cos^2 \phi \sin \phi d\theta d\phi \\ &= 2\pi R^3 \int_0^{\pi/2} \cos^2 \phi \sin \phi d\phi = 2\pi R^3 \left[-\frac{1}{3} \cos^3 \phi \right]_0^{\pi/2} = \frac{2\pi R^3}{3} \end{aligned}$$

Example 95. Find the flux

$$\Phi = \oiint_S \vec{F} \cdot \vec{n} dS$$

of $\vec{F} = x\vec{i} + y\vec{j} + 3\vec{k}$ through the surface of the paraboloid $z = x^2 + y^2$ bounded by $z = 0$ and $z = 4$.

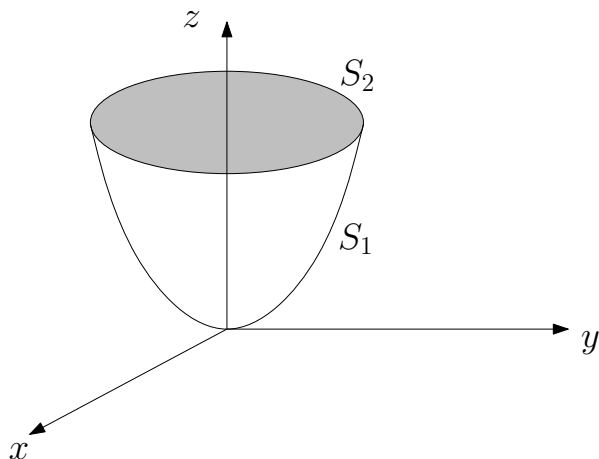


Figure 108: The given surface is divided in two parts: the paraboloid S_1 and the cap S_2 (shaded).

We can divide the surface S in two parts: the paraboloid surface S_1 and the cap S_2 as shown in figure 108. Thus, we can write the flux for the cap as:

$$\iint_{S_2} \vec{F} \cdot \vec{n} \, dS = 3 \text{ Area} = 12\pi$$

We will use the equation for special case 1 and write $z = x^2 + y^2$. Thus

$$\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right) = (-2x, -2y, 1)$$

We assume that the normal \vec{n} points outward to the surface and hence we have the normal as $\vec{n} = (2x, 2y, -1)$. Note the sign change! Thus we have:

$$\begin{aligned} \iint_{S_1} \vec{F} \cdot \vec{n} \, dS &= \iint \vec{F} \cdot (2x, 2y, -1) \, dx \, dy = \iint (2x^2 + 2y^2 - 3) \, dx \, dy \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^2 (2r^2 - 3) r \, dr \, d\theta = 4\pi \end{aligned}$$

Summing the flux over surfaces S_1 and S_2 , we obtain the total flux as $\Phi = 4\pi + 12\pi = 16\pi$.

23.2 Applications

23.2.1 Heat flow

Heat flow refers to the amount of energy that is transferred in the form of heat. We can compute the heat flow \vec{q} in terms of the temperature T as:

$$\vec{q} = -K\nabla T$$

where K is a constant. Thus the rate of heat flow out of some region R is defined as:

$$\iint_S \vec{q} \cdot \vec{n} \, dS = -K \iint_S \nabla T \cdot \vec{n} \, dS$$

where S is the boundary of the region R .

23.2.2 Fluid flow

Let's say we have a region T bounded by the surface S . The fluid flow of the velocity field \vec{v} out of the region T is given by:

$$\iint_S \vec{v} \cdot \vec{n} dS$$

If we have an incompressible fluid, then

$$\iint_S \vec{v} \cdot \vec{n} dS = 0$$

23.2.3 Force fields

The flux of an electric field \vec{E} through a region S is given by

$$\iint_S \vec{E} \cdot \vec{n} dS = \frac{Q}{\epsilon_0}$$

where Q is the total charge in the region and ϵ_0 is the dielectric constant.

Similarly, if we have the gravitational field \vec{F} , then the flux through a region S is given by

$$\iint_S \vec{F} \cdot \vec{n} dS = -4\pi GM$$

23.3 Rocket nose cone

A rocket nose (shown in figure 109) has to be designed such that it generates the lowest amount of heat, which is proportional to the amount of drag it generates.

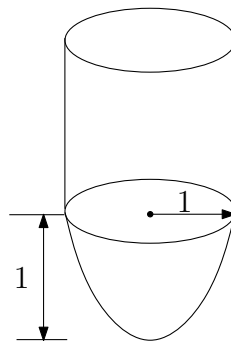


Figure 109: A rocket nose cone

Let's say we have 4 candidates for the nose shape, as shown in figure 110.



Figure 110: Four candidate shapes for the nose cone

We will assume that the height of the nose is 1 and the width is 2. The air resistance is given by $2\pi\delta Rv^2$, where δ is the air density, R is the drag coefficient and v is the velocity of the rocket. For a surface of revolution (such as all our candidates), R can be approximated as:

$$R = \frac{1}{\pi} \iint_S \cos^3 \phi \, dS$$

where ϕ is the angle of the curve which is revolved to get the surface as shown in figure 111.

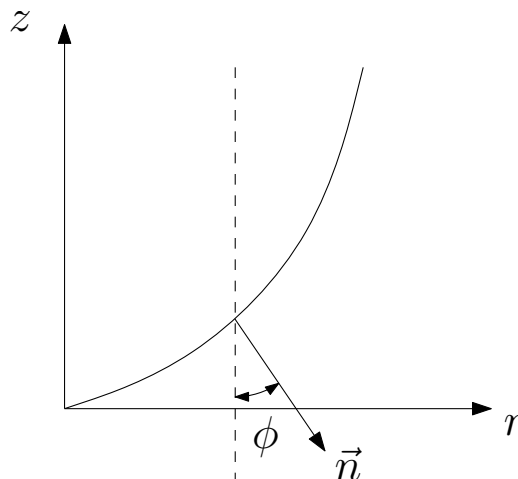


Figure 111: Defining the angle ϕ for a surface of revolution

We can parameterize the surface as $x = r \cos \theta$, $y = r \sin \theta$ and $z = z(r)$. Then we can write the normal as:

$$\vec{n} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial r} = (r \cos \theta \, z', r \sin \theta \, z', -r)$$

This gives us $|\vec{n}| = r\sqrt{1 + (z')^2}$, and consequently:

$$\cos \phi = \frac{-\vec{n} \cdot \vec{k}}{|\vec{n}|} = \frac{r}{r\sqrt{1 + (z')^2}} = \frac{1}{\sqrt{1 + (z')^2}}$$

Thus,

$$\begin{aligned} R &= \frac{1}{\pi} \iint_S \cos^3 \phi \, dS = \frac{1}{\pi} \iint_S \frac{1}{(1 + (z')^2)^{3/2}} r \sqrt{1 + (z')^2} \, dr \, d\theta \\ &= \frac{1}{\pi} \iint \frac{r}{1 + (z')^2} \, dr \, d\theta = 2 \int_0^1 \frac{r}{1 + (z')^2} \, dr \end{aligned}$$

Now, we can compute the value of the drag coefficient R for each of the shapes to get:

1. Pointy nose: $z = r$ and we can compute $R = 1/2$.
2. Flat tipped nose: for this shapes, we can compute $R \approx 0.38$ (if we optimize the angle of the oblique segment).
3. Round nose: $z = 1 - \sqrt{1 - r^2}$ and we can compute $R = 1/2$.
4. Parabolic nose: $z = r^2$ and can compute $R = \frac{\ln 5}{4} \approx 0.4$.

Thus, the flat nose offers the least resistance which is not at all obvious in the first glance!

24 Stokes' Theorem

Recall the circulation-curl form of the Green's theorem:

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Stokes' theorem is an extension of the circulation-curl form of Green's theorem to three dimensions.

24.1 Curl

Let's say we have a vector field $\vec{F}(x, y, z) = (M, N, P)$. Then, we can define the curl (denoted by $\nabla \times$) of the vector field \vec{F} as:

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \vec{k}$$

A useful way to remember the formula for curl is to think of it in terms of the following determinant:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Note that this is a special determinant as it has operators in the second row instead of numerical quantities.

24.2 Stokes' Theorem

The circulation-curl form of Green's theorem can be written in terms of the curl as:

$$\oint_C (\vec{F} \cdot \vec{T}) ds = \iint_R (\nabla \times \vec{F}) \cdot \vec{k} dx dy$$

Stokes' theorem can be written as this more general form which holds for three dimensions as well:

$$\boxed{\oint_C (\vec{F} \cdot \vec{T}) ds = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dx dy}$$

where S is the surface over which the vector field \vec{F} is defined. Note that the orientation of the normal must be such that $\vec{n} \times \vec{T}$ points towards the surface as shown in figure 112.

24.2.1 Proof of Stokes' theorem

We will give here a partial proof of Stokes' theorem, for the case when the surface S is expressible in the form $z = f(x, y)$. We will only prove that

$$\oint_C M dx = \iint_S \left(\frac{\partial M}{\partial z} \vec{j} - \frac{\partial M}{\partial y} \vec{k} \right) \cdot \vec{n} dx dy$$

which is the part of Stokes' theorem in which we take all the terms with M occurring on the left and the right hand sides. You should be able to prove on your own the remaining two parts for components N and P , as the proof is along similar lines.

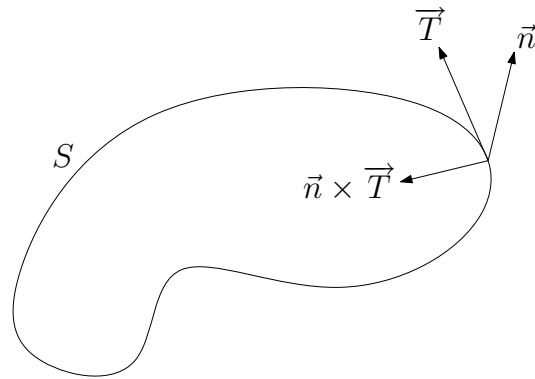


Figure 112: The correct orientation of the normal vector

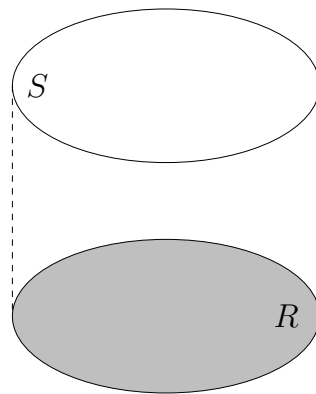


Figure 113: Proving Stokes' theorem

The surface S and its projection R on the xy -plane are shown in figure 113. First, we see that M is a function of $(x, y, z(x, y))$. Hence we can apply Green's theorem to write:

$$\oint_C M dx = - \iint_R \frac{\partial}{\partial y} [M(x, y, z(x, y))] dx dy = - \iint_R \left(\frac{\partial M}{\partial y} + \frac{\partial M}{\partial z} \frac{\partial z}{\partial y} \right) dx dy$$

Now, we recall special case 1 of the surface integrals to write:

$$\iint_S \left(\frac{\partial M}{\partial z} \vec{j} - \frac{\partial M}{\partial y} \vec{k} \right) \cdot \vec{n} dx dy = - \iint_R \left(\frac{\partial M}{\partial y} + \frac{\partial M}{\partial z} \frac{\partial z}{\partial y} \right) dx dy$$

Since these are equal, we have proven

$$\oint_C M dx = \iint_S \left(\frac{\partial M}{\partial z} \vec{j} - \frac{\partial M}{\partial y} \vec{k} \right) \cdot \vec{n} dx dy$$

Example 96. Use Stokes' theorem to evaluate the line integral of $\vec{F} = y\vec{i} - x\vec{j}$ over the circle $x^2 + y^2 = 4$ using different methods.

- Using a parametrization of the circle: we can parameterize the circle as: $x = 2 \cos \theta$ and $y = 2 \sin \theta$. This gives us $dx = -2 \sin \theta d\theta$ and $dy = 2 \cos \theta d\theta$. Note that we can express the line integral as:

$$\int (\vec{F} \cdot \vec{T}) ds = \int \vec{F} \cdot d\vec{r}$$

We can compute

$$\vec{F} \cdot d\vec{r} = (2 \sin \theta, -2 \cos \theta) \cdot (-2 \sin \theta, 2 \cos \theta) d\theta = -4 d\theta$$

This gives us the line integral as

$$\int \vec{F} \cdot d\vec{r} = -4 \int_0^{2\pi} d\theta = -8\pi$$

- Using the disk in the xy -plane:

$$\int (\vec{F} \cdot \vec{T}) ds = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right) dx dy = -2 \pi (2)^2 = -8\pi$$

- Using a surface $z = f(x, y)$ with boundary $x^2 + y^2 = 4$. Recall that we can write

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dx dy = \iint \left(-M \frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} + P \right) dx dy$$

This gives us:

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dx dy = (-2) \iint dx dy = -8\pi$$

Example 97. Using Stokes' theorem, evaluate the line integral $\oint \vec{F} \cdot d\vec{r}$ if $\vec{F} = xz\vec{i} + xy\vec{j} + 3xz\vec{k}$ over the curve defined by the portion of the plane $2x + y + z = 2$ in the first octant, traversed counterclockwise.

We can compute the curl as follows:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix} = (x - 3z)\vec{j} + y\vec{k}$$

Note that our surface is given by the plane $2x + y + z = 2$ and hence we can write $z = 2 - 2x - y$. The normal vector is given by

$$\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = (2, 1, 1)$$

We need to plot this as shown in figure 114 to determine the correct orientation of the normal vector.

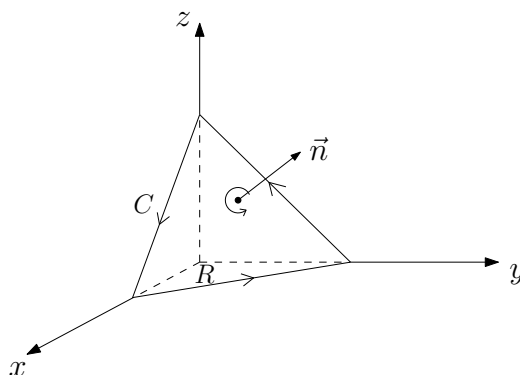


Figure 114: Checking the orientation of the normal vector

Note that $M = 0$, $N = x - 3z = x - 3(2 - 2x - y) = 7x + 3y - 6$ and $P = y$. Now, we will use the formula for special case 2 to get:

$$\begin{aligned} \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS &= \iint_R ((x - 3z) + y) \, dx \, dy \\ &= \int_0^1 \left[\int_0^{2-2x} (-6 + 7x + 4y) \, dy \right] dx \\ &= \int_0^1 ((-6 + 7x)(2 - 2x) + 2(2 - 2x)^2) \, dx \\ &= \int_0^1 (-12 + 26x - 14x^2 + 8 - 16x + 8x^2) \, dx \\ &= -4 + 5 - 2 = -1 \end{aligned}$$

24.3 Applications

24.3.1 Ampere's circuital law

Let's say we have a magnetic field \vec{B} generated by a wire which has a current I flowing through it, as shown in figure 115. This field is given by

$$\int \vec{B} \cdot \vec{T} \, ds = \mu_0 I$$

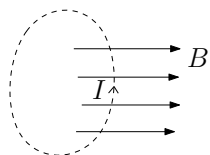


Figure 115: Magnetic field generated by a current in a wire

We define the quantity \vec{J} which is the current density field whose magnitude gives the current per unit area and whose direction gives the direction of current. Hence, we have

$$I = \iint_R \vec{J} \cdot \vec{n} \, ds$$

Let us apply Stokes' theorem to the above integral to get:

$$\oint_C \vec{B} \cdot \vec{T} \, ds = \iint_S (\nabla \times \vec{B}) \cdot \vec{n} \, dS$$

Thus, we get that

$$\iint_S (\nabla \times \vec{B}) \cdot \vec{n} \, dS = \iint_R (\mu_0 \vec{J}) \cdot \vec{n} \, dS$$

Since this equation holds for an arbitrary surface, we must have

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

which is the differential form of Ampere's law.

24.3.2 Faraday's law of induction

Faraday's law of induction states that the change in magnetic flux is the same as the electromagnetic force around a boundary. This can be stated mathematically as:

$$\frac{d\Phi}{dt} = - \oint \vec{E} \cdot d\vec{s}$$

We can apply Stokes' theorem to this integral to get

$$\oint \vec{E} \cdot \vec{T} \, ds = \iint (\nabla \times \vec{E}) \cdot \vec{n} \, dS$$

Also, we know that

$$\Phi = \iint \vec{B} \cdot \vec{n} \, dS$$

Differentiating this expression and equating with the above, we get

$$\iint \frac{d\vec{B}}{dt} \cdot \vec{n} \, dS = - \iint (\nabla \times \vec{E}) \cdot \vec{n} \, dS$$

Since this is true for any surface, we have

$$\boxed{\frac{d\vec{B}}{dt} = -\nabla \times \vec{E}}$$

which is the differential form of Faraday's law.

25 Divergence Theorem

The divergence theorem is the 3D extension of the flux-divergence form of Green's theorem, similar to Stokes' theorem. Recall that the flux-divergence form of Green's theorem is given by:

$$\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy$$

Note that we can write

$$\oint_C M dy - N dx = \oint_C \vec{F} \cdot \vec{n} ds$$

which is the left hand side of Green's theorem. Also, we can write the right hand side as a special case in 2D of a 3D equation:

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Green's theorem can be expressed in the more general form of the divergence theorem which is valid in 3D:

$$\boxed{\iint_S \vec{F} \cdot \vec{n} ds = \iiint_D \nabla \cdot \vec{F} dV}$$

where S is a closed surface which forms a simple solid region D . Also, we assume that \vec{F} is differentiable in D and \vec{n} points out of the region.

Example 98. Calculate the flux $\iint_C \vec{F} \cdot \vec{n} ds$ of $\vec{F} = xy\vec{i} + yz\vec{j} + xz\vec{k}$ through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$. Then verify your result using the divergence theorem.

We need to compute the flux through each of the 6 faces of the cube, as follows:

1. $x = 0$: we have $\vec{n} = -\vec{j}$ and $\vec{F} = xz\vec{k}$, which implies that $\vec{F} \cdot \vec{n} = 0$ and hence flux is 0.
2. $y = 0$: we have $\vec{n} = -\vec{i}$ and $\vec{F} = yz\vec{j}$, which implies that $\vec{F} \cdot \vec{n} = 0$ and hence flux is 0.
3. $z = 0$: we have $\vec{n} = -\vec{k}$ and $\vec{F} = xy\vec{i}$, which implies that $\vec{F} \cdot \vec{n} = 0$ and hence flux is 0.
4. $y = 1$: we have $\vec{n} = \vec{j}$ and $\vec{F} = x\vec{i} + z\vec{j} + xz\vec{k}$. Hence the flux is given by:

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_S \vec{F} \cdot \vec{n} dx dz = \int_0^1 \int_0^1 z dx dz = \frac{1}{2}$$

5. $x = 1$: we have $\vec{n} = \vec{i}$ and $\vec{F} = y\vec{i} + yz\vec{j} + z\vec{k}$. Hence the flux is given by:

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_S \vec{F} \cdot \vec{n} dx dz = \int_0^1 \int_0^1 y dy dz = \frac{1}{2}$$

6. $z = 1$: we have $\vec{n} = \vec{k}$ and $\vec{F} = xy\vec{i} + y\vec{j} + x\vec{k}$. Hence the flux is given by:

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_S \vec{F} \cdot \vec{n} dx dy = \int_0^1 \int_0^1 x dx dy = \frac{1}{2}$$

The flux over the entire surface is the sum of fluxes through all the individual faces and so $\Phi = \frac{3}{2}$. Now, the divergence is given by:

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

Hence, we can compute:

$$\iiint_D \nabla \cdot \vec{F} dV = \int_0^1 \int_0^1 \int_0^1 (x + y + z) dx dy dz = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

This agrees with the divergence theorem.

Example 99. S is the boundary surface of the region T defined by $z = 0$, $y = 0$, $y = 2$ and the parabolic cylinder $z = 1 - x^2$, as shown in figure 116. Compute the flux of \vec{F} through S where $\vec{F} = (x + \cos y)\vec{i} + (y + \sin z)\vec{j} + (z + e^x)\vec{k}$.

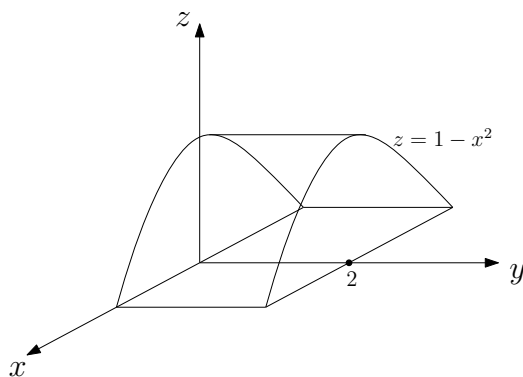


Figure 116: A parabolic cylinder bounded by planes

We can compute the divergence as:

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{\partial}{\partial x}(x + \cos y) + \frac{\partial}{\partial y}(y + \sin z) + \frac{\partial}{\partial z}(z + e^x) = 3$$

Hence, we can compute the integral of the divergence as:

$$\iiint_D \nabla \cdot \vec{F} dV = \iiint_D 3 dx dy dz = 3(\text{Volume}) = 6(\text{Area}) = 6 \int_{-1}^1 (1 - x^2) dx = 8$$

which is the flux using the divergence theorem.

25.1 A note about differentiability

Note that a key assumption of the theorem is that \vec{F} must be differentiable *everywhere* inside the domain. So what if the vector field is discontinuous/has a singularity at some point? Let us say that the vector field \vec{F} has a singularity at 0. An example of such a field is:

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$$

We can apply the divergence theorem to any region in which \vec{F} has a singularity at one point by excluding a small region around the singularity, as shown in the figure 117. This means that our surface S is now divided into 2 parts: S_1 and S_2 and we can apply Stokes' theorem as usual.

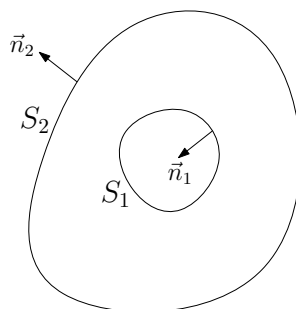


Figure 117: A trick to get around singularity of a vector field

Example 100. Show that for any surface S surrounding a charge Q , the electric field $\vec{E} = \frac{\epsilon_0 Q}{|\vec{r}|^3} \vec{r}$ satisfies

$$\oiint_S \vec{E} \cdot \vec{n} \, dS = 4\pi\epsilon_0 Q$$

Let's say we have a charge Q sitting at the origin in a electric field \vec{E} . Let's compute the flux through a sphere of radius a about the origin:

$$\oiint_S \vec{E} \cdot \vec{n} \, dS = \oiint_S \frac{\epsilon_0 Q}{|\vec{r}|^3} \frac{\vec{r}}{a} \cdot \frac{\vec{r}}{a} \, dS = \oiint_S \frac{\epsilon_0 Q}{a^2} \, dS = \frac{\epsilon_0 Q}{a^2} 4\pi a^2 = 4\pi\epsilon_0 Q$$

which is independent of a !

Now, let's say we have an arbitrary surface instead of the sphere which includes the origin. We can exclude the origin by creating a small sphere around it, as suggested in the note above. Hence, we can apply the divergence theorem to get:

$$\iint_{S_1} \vec{E} \cdot \vec{n} \, dS + \iint_{S_2} \vec{E} \cdot \vec{n} \, dS = \iiint_D \nabla \cdot \vec{E} \, dV$$

But, $\nabla \cdot \vec{E} = 0$, and thus, we have:

$$\iint_{S_1} \vec{E} \cdot \vec{n} \, dS = - \iint_{S_2} \vec{E} \cdot \vec{n} \, dS = 4\pi\epsilon_0 Q$$

for an arbitrary surface S_1 ! This is analogous to a calculation we did with Green's theorem previously.

Example 101. Show that the electric field $\vec{E} = \frac{\epsilon_0 Q}{|\vec{r}|^3} \vec{r}$ satisfies $\nabla \cdot \vec{E} = 4\pi\epsilon_0 \rho$, where ρ is the charge density per unit volume.

We can compute the flux as:

$$\oiint_S \vec{E} \cdot \vec{n} \, dS = 4\pi\epsilon_0 Q = 4\pi\epsilon_0 \iiint_D \rho \, dV = \iiint_D (4\pi\epsilon_0 \rho) \, dV = \iiint_D \nabla \cdot \vec{E} \, dV$$

But this is true for an arbitrary volume D and hence we must have $\nabla \cdot \vec{E} = 4\pi\epsilon_0 \rho$.

25.2 Continuity equation

Show that any fluid flow, with density δ , satisfies:

$$\frac{\partial \delta}{\partial t} + \nabla \cdot (\delta \vec{v}) = 0$$

The mass is given by

$$M = \iiint_D \delta \, dV$$

The mass that is leaving the volume per unit time can be computed as:

$$\frac{dM}{dt} = - \iint_S (\delta \vec{v}) \cdot \vec{n} \, dS = - \iiint_D \nabla \cdot (\delta \vec{v}) \, dV$$

Also, we must have

$$\frac{dM}{dt} = \frac{d}{dt} \iiint_D \delta \, dV = \iiint_D \frac{\partial \delta}{\partial t} \, dV$$

Since these two equations hold for arbitrary surfaces, we must have:

$$\frac{\partial \delta}{\partial t} + \nabla \cdot (\delta \vec{v}) = 0$$

This equation is called the **continuity equation** for compressible flows.

25.3 Archimedes' law of buoyancy

The pressure on a body submerged in a fluid is given by $P = -\rho g z$ and the buoyant force is given by

$$\vec{B} = - \iint_S P \vec{n} \, dS$$

We are going to prove the law of buoyancy using the divergence theorem.

If we have a force field \vec{F} of the form $\vec{F} = (f, 0, 0)$, then the divergence theorem gives us:

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_D \nabla \cdot \vec{F} \, dV$$

If the normal vector is given by $\vec{n} = (n_1, n_2, n_3)$, then this equation becomes:

$$\iint_S f n_1 \, dS = \iiint_D \frac{\partial f}{\partial x} \, dV$$

Similarly, we can apply the divergence theorem for $\vec{F} = (0, f, 0)$ and $\vec{F} = (0, 0, f)$, and then sum the resulting equations with the one above to get:

$$\iint_S f \vec{n} \, dS = \iiint_D \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \right) \, dV$$

Apply this result to $f = -P$:

$$\vec{B} = - \iint_S P \vec{n} \, dS = g \left(\iiint_D \rho \, dV \right) \vec{k} = Mg \vec{k}$$

which is the weight of the fluid displaced by the body.

26 Conservative and irrotational fields

Recall that a force field \vec{F} is conservative if the work done $W = \int_A^B \vec{F} \cdot \vec{T} ds$ is independent of the path taken from A to B . This is equivalent to saying that a force field \vec{F} is conservative *if and only if* there exists a potential function ϕ such that $\vec{F} = \nabla\phi$. Here, we show that there is yet another way of showing that a vector field \vec{F} is conservative, and this is that the curl of a conservative field is 0, i.e., $\nabla \times \vec{F} = 0$. This statement works in the reverse direction as well, i.e., if the curl of a force field is 0, i.e., $\nabla \times \vec{F} = 0$, then \vec{F} is conservative. We can prove this statement as follows:

Proof. First, let us assume that \vec{F} is conservative and show that the curl of \vec{F} is 0. Since \vec{F} is conservative, we have a potential function ϕ such that $\vec{F} = \nabla\phi$. For a vector field of this form, we can explicitly verify that $\nabla \times (\nabla\phi) = 0$. For example, the x component of this cross product is given by:

$$\frac{\partial}{\partial y} \left(\frac{\partial\phi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial\phi}{\partial y} \right) = 0$$

This proves one direction.

For the other direction, we assume that $\nabla \times \vec{F} = 0$ and show that \vec{F} is conservative. Let us look at the work done by the vector field \vec{F} while going through a closed loop from point A to B through C_1 and from B to A through C_2 , as shown in figure 118.

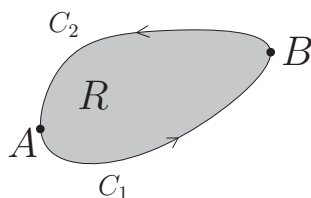


Figure 118: The closed path from a point A back to itself through B

Let the closed curve C be the union of these two curves: $C = C_1 \cup C_2$. We have the work done as:

$$W = \oint_C \vec{F} \cdot \vec{T} ds = \int_R (\nabla \times \vec{F}) \cdot \vec{k} ds = 0$$

as $\nabla \times \vec{F} = 0$. But the integral on the left hand side can be written as

$$W = \oint_C \vec{F} \cdot \vec{T} ds = \int_{C_1} \vec{F} \cdot \vec{T} ds + \int_{C_2} \vec{F} \cdot \vec{T} ds$$

This implies that:

$$\int_{C_1} \vec{F} \cdot \vec{T} ds = - \int_{C_2} \vec{F} \cdot \vec{T} ds$$

The right hand side corresponds also to an integral along a path going from A to B following the curve C_2 . This is the reverse of the path C_2 which goes from B to A and hence the negative sign in front of the integral. This equation says that the work done in moving from A to B is the same along C_1 and C_2 ; this shows that in fact it is independent of the path. We have proved that the vector field \vec{F} is conservative. \square