CME 102, Winter 2009
Ordinary Differential Equations for Engineers
Prof. Eric Darve

Lecture Notes

CME 102 is cross referenced as ENGR 155A. The topic covered in this course is Ordinary Differential Equations, or ODEs, which is a fundamental topic in all engineering branches. This course distinguishes from those in the Mathematics department as it offers two perspectives for the study of ODEs:

1. Theoretical perspective. This one is similar to that offered in the Mathematics department.

2. Use of computer software to solve ODEs. This is offered only here in CME 102, and is a very important component of the course.

We are going to use the best software available to solve ODEs, namely Matlab. We are not only going to learn how to solve ODEs with Matlab, we will also learn how to program with it. This is an essential tool for all engineers, so tutorials have been scheduled to ease the use and programming using Matlab.
Whenever we use a computational method to solve an engineering problem, we have to deal with questions such as how "good" or precise the solution is? We are thus also going to do some *Numerical Analysis*. One of the goals of numerical analysis is to answer such questions.
For more information on the administrative aspects of the course, and on its contents, please refer to the handouts distributed on the first day of class. If you lost one of them, don't worry as all handouts will be available on the web through coursework.

1 Introduction to ODEs

1.1 A cup of coffee

We are going to introduce Ordinary Differential Equations through a cup of hot coffee. You can think about this cup of coffee as an engineering project. The first question one has to answer is how to describe such cup of coffee. There are clearly several characteristics of this cup of coffee:

1. Dimensions of the cup itself.

2. Temperature of the coffee.

The first set of characteristics is fixed and will not change, and thus can be represented by a set of numbers (say the height and radius of the top and bottom of the cup).

The temperature is more complicated to describe. It is likely that the upper part of the coffee will be colder than the bottom part of it. This is due to coffee's interaction with the air. Furthermore, the temperature will change with time. A full model of this situation is given in CME 104 as this problem requires Partial Differential Equations and not only ODEs. To go back to ODEs' realm, we simply stir the coffee so that now temperature is uniform throughout the coffee.
It is important to realize that, while at a given moment we can represent coffee's temperature with a single number, this temperature still changes with time. So we are in the presence of a characteristic of our coffee that *cannot* be simply represented by a single number but instead by a function of time $T(t)$. Now, the first natural approach to describing the coffee's temperature is to describe the *rate of change* in temperature as a function of time

$$\frac{dT}{dt}$$

Now assume that the room is at a temperature T_a of, say, 70 degrees Fahrenheit. If the coffee is at the same temperature as the room, then one would expect to have a constant coffee temperature. Given that we assumed it is hot coffee we are dealing with, we have that $T(t) > T_a$. Thus, a reasonable first approximation to the rate of change in temperature is that it is proportional to the difference in room temperature and the coffee's current temperature. Further, if the coffee is hotter than the room, we know that its temperature will decrease with time. We have then the following model for the coffee's temperature:

$$\frac{dT}{dt} = -k(T - T_a). \tag{1}$$

where k is a positive constant that depends on the cup's materials.
This is a simple model that delivers very good results. This is also our first ODE! It is an ODE because it is an equation that involves a function, say T, and its derivatives.
We now solve this ODE by direct integration:

$$\frac{dT}{T - T_a} = -k\,dt \text{ which yields } \log\left(|T - Ta|\right) = -kt + c$$

where c is a constant that arises from integrating the equation. It is very important not to forget that constant! We will later see where it comes from and it is useful.
We now take the exponential and get $|T - T_a| = \exp(-kt + c) = \exp(c)\exp(-kt)$. But c is an arbitrary constant, hence $\exp(c)$ is also an arbitrary constant. If we define $c_0 = \exp(c)$, we get $T - T_a = c_0 \exp(-kt)$, which yields the following equation for the temperature

$$T(t) = T_a + c_0 \exp(-kt). \tag{2}$$

This is called the **general solution** to the ODE 1. We call it a general solution because it involves an arbitrary number c_0. We will see later on that the initial condition (that is the value $T(0)$) is used to determine c_0.

Let us now verify that the general solution indeed satisfies the ODE. We start by calculating the derivative of T using the expression from the general solution:

$$\frac{dT}{dt} = c_0(-k)\exp(-kt) = -k\left[c_0\exp(-kt)\right].$$

But, from the general solution we also have that $T - T_a = c_0\exp(-kt)$. Thus we conclude that

$$\frac{dT}{dt} = -k\left[T - T_a\right]$$

as claimed.

We now define a **particular solution** where c_0 is going to assume a specific value. But how can we select the value of c_0? If we go back to our coffee problem, we assume that we know the initial temperature of the coffee, say 190 degrees Fahrenheit. In other words, assume that

$$T(0) = T_0. \tag{3}$$

Such condition is called the *initial condition*. The particular solution we are looking for is one that satisfies *simultaneously* both the ODE 1 and the initial condition 3.

Any well posed system has a differential equation as well as a set of initial conditions. Initial conditions are needed to *uniquely* characterize the solution to the differential equation. Let us now calculate the constant c_0:

$$T(0) = T_a + c_0\exp(-k0) = T_a + c_0 \text{ thus we have that } c_0 = T_0 - T_a.$$

We can now write the particular solution to the cup of coffee problem

$$T(t) = T_a + (T_0 - T_a)\exp(-kt). \tag{4}$$

Here is a plot of how the temperature would look like when initially the coffee is at 190 degrees Fahrenheit, and the room is at 70 degrees Fahrenheit.

What we just solved is what is called an **Initial Value Problem**, which is a combination of an ODE and a set of initial conditions. There are other ways to model an engineering problem, namely *Boundary Value Problems* which are a combination of an ODE and a set of conditions that hold *at both ends of the interval*. We are only going to focus on Initial Value problems.

Here is the matlab code used for this plot:

```
>> t = linspace(0,20,100);
>> Ta = 70; T0 = 190; k = 1/20;
>> plot(t,Ta+(T0-Ta)*exp(-k*t))
>> xlabel('Time [min]');
>> ylabel('Temperature [F]')
```

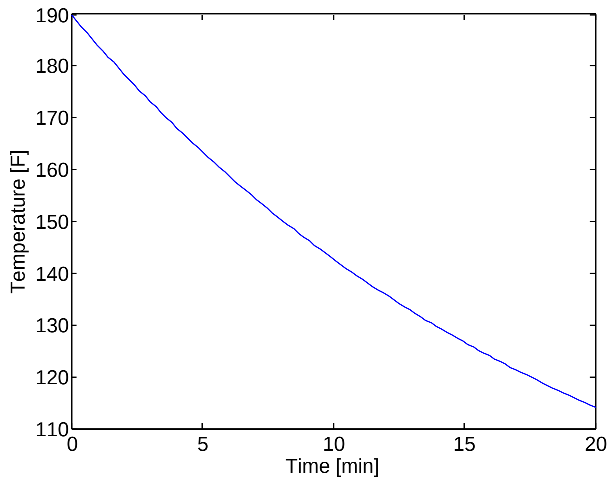


Figure 1: Evolution of the coffee's temperature. $T_a = 70$, $T_0 = 900$.

1.2 Partial Differential Equations

Let us now contrast Partial Differential Equations and Ordinary Differential Equations. As we mentioned before, ODEs constitute the basis for the theory of differential equations. Many methods used to solve Partial Differential Equations (PDEs) sometimes reduce to solving an ODE. Recall that an ODE is an equation involving a function, say f, and its derivatives, $\frac{df}{dx}$, $\frac{d^2f}{dx^2}$, etc.

A PDE is a differential equation that involves a function that depends on several variables and its derivatives. For instance, the function can be the pressure in the room $p(x, y, z)$ that depends on space, or on space and time $p(x, y, z, t)$. We now give some examples of PDEs in physics.

Example 1 Wave Equation The wave equation is used in many different physical systems. We are going to focus on a vibrating string of, say, a guitar.

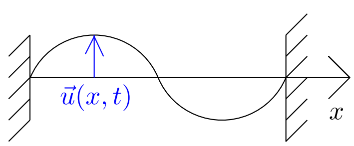


Figure 2: Guitar String

Let $u(x, t)$ be the displacement of the string at position x and time t. The equation satisfied by such system is

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}. \quad (5)$$

This is a widely used model in physics. For instance, it is used to model sound waves, telluric waves and even light waves.

PDEs are studied in CME 104, which is offered next quarter. We now go back to the focus of this problem, ODEs. We are going to draw our example from both mechanics and electrical engineering.

1.3 Examples

Example 2 Pendulum We are going to provide a model to represent the system. We will not solve now the resulting equation, though we will later in the course.

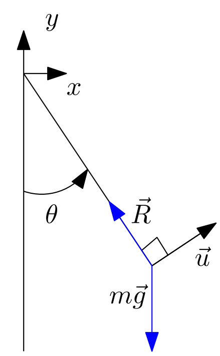


Figure 3: Pendulum

We call θ the angle that the pendulum does with respect to the vertical direction. Let l be the length of the string linking the mass m of the pendulum to the ceiling. We assume that both l and m are constant. From figure 3, we see that if we call (x, y) the spatial coordinates of the pendulum's mass, we have the following relations between x, y, l, and θ.

$$x = l\sin(\theta)$$
$$y = -l\cos(\theta)$$

Thus we have the following expressions for the time derivatives of x and y:

$$x' = l\theta'\cos(\theta)$$
$$x'' = l\theta''\cos(\theta) - l(\theta')^2\sin(\theta)$$
$$y' = l\theta'\sin(\theta)$$
$$y'' = l\theta''\sin(\theta) + l(\theta')^2\cos(\theta)$$

Let us now describe the forces acting on the pendulum that we will consider in our model.

- Gravity $m\vec{g}$,
- tension $\vec{R}$,

We will not consider friction in our model.
Let us call $\vec{u}$ a unit vector orthogonal to the pendulum's string. We can thus see that $\vec{u} = (\cos(\theta), \sin(\theta))$. We start by writing Newton's second law $\vec{F} = m\vec{a}$ and we project it

into $\vec{u}$:

$$\begin{aligned}\vec{F} &= m\vec{a} \\ \vec{F}\cdot\vec{u} &= m\vec{a}\cdot\vec{u} \\ \left(\vec{R}+m\vec{g}\right)\cdot\vec{u} &= m\vec{a}\cdot\vec{u} \\ \vec{R}\cdot\vec{u} - mg\sin(\theta) &= m(x''\cos(\theta)+y''\sin(\theta)) \\ -mg\sin(\theta) &= m(x''\cos(\theta)+y''\sin(\theta))\end{aligned}$$

where $\vec{R}\cdot\vec{u}=0$ because $\vec{u}$ is orthogonal to the pendulum string, and $\vec{R}$ is along the pendulum string's direction.
From the expressions of the derivatives of x and y with respect to θ, we get

$$\begin{aligned}x''\cos(\theta)+y''\sin(\theta) &= l\theta''\cos(\theta)^2 - l(\theta')^2\sin(\theta)\cos(\theta) + \\ &\quad l\theta''\sin(\theta)^2 + l(\theta')^2\cos(\theta)\sin(\theta) \\ &= l\theta''(\cos(\theta)^2+\sin(\theta)^2) \\ &= l\theta''\end{aligned}$$

Thus we have the following ODE describing the pendulum's motion

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin(\theta)=0. \tag{6}$$

This equation is different from that of the temperature in various ways. First off, the highest order derivative involved in the equation is the second derivative of θ with respect to time. Recall that the equation for the temperature involved only the first derivative of the temperature with respect to time.
We call such an equation a *second order* differential equation. The temperature's equation is a *first order* differential equation. Second, we see that the pendulum's equation involves $\sin(\theta)$. It is thus a *non-linear* equation. Recall that the temperature's equation was linear.
Those two differences are very important. We can solve a lot of first order differential equations whether linear or not. For second order differential equations, it is very difficult to solve non-linear differential equations.
For the pendulum's example, an extra assumption allows us to solve the non-linearity problem. If we assume that the angle θ is *small*, then we can make the approximation $\sin(\theta)\approx\theta$. This transforms equation 6 into a second order linear differential equation

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\theta=0. \tag{7}$$

This equation is easy to solve. We will come back to this later in the course.
Remark: Is it reasonable to neglect friction? If we consider the system for a short period of time, the force generated by friction is going to be negligible with respect to other forces

acting on the system. If we consider the system for a long period of time, we need to characterize the equilibrium behavior of the system. There friction is necessary.
Example 3 Mass Spring System This is a very important example as many engineering and physical systems can be modeled as arrays of interconnected springs. We study the simplest of those models, namely a single spring connected to a mass. Again, we neglect friction.

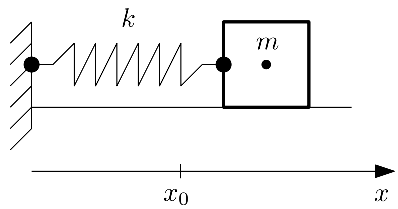


Figure 4: Mass Spring System

We call k the stiffness of the spring, that we will consider constant. Call x the position of the center of mass, and let x_0 be the position of the center of mass when the system is at rest.
Here the second Newton's law is very simple to write as the displacement of the system is only possible along the x axis. The only force acting along that axis is the force due to the spring, which is $-k(x-x_0)$. Thus, we get the following differential equation $mx'' = -k(x-x_0)$, which can be rewritten as

$$\frac{d^2x}{dt^2} + \frac{k}{m}(x-x_0) = 0.$$

We can further simplify the above equation by *changing variables*. Here the change of variables is a simple shift of x. Thus, if we set $y = x - x_0$, we get

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0. \tag{8}$$

This is a second order, linear differential equation. When the right hand side of the equation is zero, the equation is called a *homogeneous* equation. It is easy to see that this equation and equation 7 will have the same set of solutions.
Throughout the course, we will see several examples where a change of variables will prove useful in solving a differential equation.

2 Direction Field Method

The Direction Field Method is a geometrical method that yields qualitative but not quantitative properties of the differential equation studied. We start by an example from mechanics.
Example 4 Object Falling Through Atmosphere Assume a body of mass m is falling through the atmosphere. In this example we will include friction.
We assume that the force from friction is proportional to the body's speed $\vec{v}$, and write it as $-\gamma\vec{v}$ for some positive constant γ. We can now write Newton's second law $m\vec{g} - \gamma\vec{v} = m\vec{a}$.

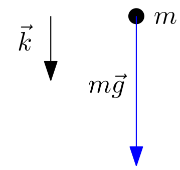


Figure 5: Falling Body

But the acceleration $\vec{a}$ is simply the derivative of the speed $\vec{v}$. If we write the speed $\vec{v} = u\vec{k}$, and we project Newton's second law into $\vec{k}$, we get

$$mg - \gamma u = m\frac{du}{dt}$$

which is, like for the temperature, a first order linear ODE.
If we further assume that $g = 9.8$ m/sec^2, and that $\gamma/m = .2$, we get the following equation

$$\frac{du}{dt} = 9.8 - \frac{u}{5}. \tag{9}$$

The method of Directional Field allows us to calculate qualitative properties of this equation, such as the *terminal velocity*. The terminal velocity is the asymptotic value the velocity the body will attain while falling through the atmosphere.
From the equation 9 we see that if we are given the body's velocity at time t, $u(t)$, we can calculate its rate of change $\frac{du}{dt}(t)$. For example, if $u = 40$, we can see that

$$\frac{du}{dt} = 9.8 - \frac{40}{5} = 9.8 - 8 = 1.8$$

Now assume we have an arbitrary ODE. We write it in the following general form:

$$\frac{dy}{dx} = f(x, y) \tag{10}$$

for some function f. We can see that the same method applies to this case: given x and y, we can calculate the slope of the solution at that point. Figure 6 illustrates the previous point.
In figure 6, the solution at (x_1, y_1) must be tangential to the short segment.
For our example of the falling object, we can draw, in the (t, u) plane, short line segments representing the tangents to the solution at those points. In this case, the slope at (t, u) is given by $9.8 - \frac{u}{5}$.
This figure was created using the following matlab code:

```
[x,y] = meshgrid(0:.75:10,40:1:60); % Creates a grid of points in the
                                    % xy plane
lineal_x = ones(size(x));
lineal_y = 9.8-y/5; % df/dy
```

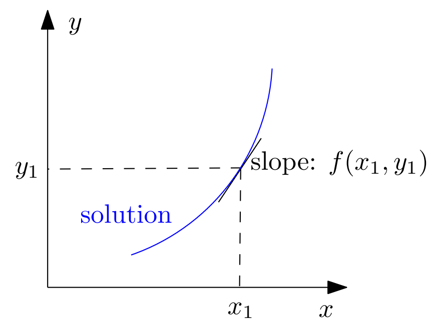


Figure 6: Direction Fields Method Illustration

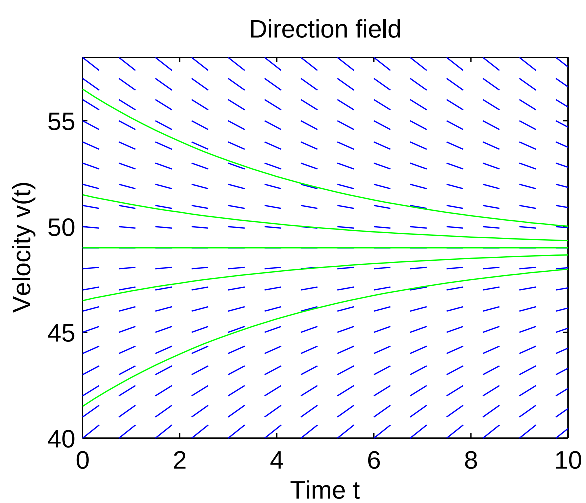


Figure 7: Example of Direction Fields Method

```
quiver(x,y,lineal_x,lineal_y,0.7,'.') % Draws the lineal elements

hold on
for k=1:4
    [x0,y0] = ode45(@(x,y) 9.8-y/5,[0 10],41.5+(k-1)*5);
    % Calculates solution
    plot(x0,y0,'g') % Plots solution
end
plot(x0,49*ones(size(x0)),'g') % Plots a line through 49

axis([0 10 40 58])
xlabel('Time t')
ylabel('Velocity v(t)')
title('Direction field')
hold off
```

We can now draw qualitative conclusions about the solution from this graph. For instance, we can see that all solutions (green curves on figure 7) converge, as t goes to infinity, to a fixed constant (here 49). Moreover, we can see that the function $u = 49$ is a solution to the ODE. We can easily check that as $9.8 - 49/5 = 0$ and, for a constant function, $\frac{du}{dt} = 0$. Such solutions are called by various names: *equilibrium points, critical points* or *steady states*.
But we can say more than that. We can see that if $u < 49$, then u's value increases and tends to 49. Similarly, if $u > 49$, u's value decreases and tends again to 49. Given that 49 is an equilibrium point and that, if slightly perturbed from the equilibrium, the system tends to go back to this equilibrium point, we say that 49 is a *stable equilibrium point*.
The study of equilibrium points is very important. The next section expands more on that topic.

3 Equilibrium Points

Equilibrium points are also called critical points or steady states. We can define an equilibrium point as follows. We say that y_{eq} is an equilibrium point for the ODE $\frac{dy}{dx} = f(x, y)$ if $y(x) = y_{\text{eq}}$ satisfies the ODE. Thus it is sufficient for y_{eq} to be such that, for all x, $0 = f(x, y_{\text{eq}})$.
There are two types of equilibrium points. In the body falling through the atmosphere we encountered the first one, namely that of a *stable* equilibrium. The second type of equilibrium points are *unstable* equilibrium points. In the falling body through the atmosphere, there is only one stable equilibrium. As an example of an unstable equilibrium, we go back to the pendulum example.
As we can see in figure 8, the second equilibrium point is such that any perturbation from the equilibrium point, no matter how small, will make the system leave the equilibrium point.

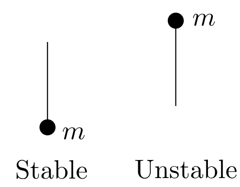


Figure 8: Equilibrium Points for the Pendulum

We can summarize our discussion of types of equilibrium with the following list. If the equilibrium point is such that after a

1. sufficiently small perturbation, the system tends back to the equilibrium point, then the equilibrium point is said to be *stable*.

2. arbitrarily small perturbation, the system leaves the equilibrium point, then the equilibrium point is said to be *unstable*.

It is important to note that systems might have only stable equilibrium points (such as the free fall system), stable and unstable ones (such as the pendulum), only unstable or no equilibrium points at all. The study of equilibrium points is very important. We will later see an illustrative example from population dynamics.

4 Solution Techniques for ODEs: Separation of Variables

Recall from equation 10 that, in its most general form, a first order ODE can be written as

$$\frac{dy}{dx} = f(x, y)$$

for some function f. We can always provide numerical solutions to such ODEs, but it is in general impossible to solve them analytically. There are some situations where analytical solutions can be given. We explore those situations in the following sections.
Assume that there exist functions $M(x)$ and $N(y)$ such that $f(x, y) = M(x)N(y)$. We then can solve those ODEs. The general technique is to put all the factors depending on y on the left hand side, and all those depending on x on the right hand side. Thus we have that equation 10 yields

$$\frac{dy}{N(y)} = M(x)dx.$$

The general solution is given by

$$\int \frac{dy}{N(y)} = \int M(x)dx + c \tag{11}$$

where c is an integration constant.

Example 5 Linear ODE Assume we want to solve $y' = -\frac{y-3}{2}$. This is a separable equation. We can set $M(x) = -1/2$ and $N(y) = y - 3$. The solution is given by

$$\begin{aligned} \int \frac{dy}{N(y)} &= \int M(x)dx + c \\ \int \frac{dy}{y-3} &= \int \left(-\frac{dx}{2}\right) + c \\ \log|y-3| &= -\frac{x}{2} + c, \text{ we take the exponential and get} \\ |y-3| &= \exp\left(-\frac{x}{2} + c\right) = e^c e^{-\frac{x}{2}}. \end{aligned}$$

Now, if we set $c_0 = e^c$ if $y - 3 > 0$, and $c_0 = -e^c$ if $y - 3 \leq 0$, we get the final solution

$$y(x) = 3 + c_0 e^{-\frac{x}{2}}.$$

Like in the free fall example, this equation has a stable equilibrium point at $y = 3$.
Let us now find a particular solution. If we set $y(x = 0) = 2$, we can now determine c_0:

$$\begin{aligned} y(0) &= 3 + c_0 e^{-0} \\ &= 3 + c_0 \\ &= 2 \end{aligned}$$

thus we have that $2 = 3 + c_0$, or $c_0 = -1$. Figure 9 is a plot of the graph of this particular solution.

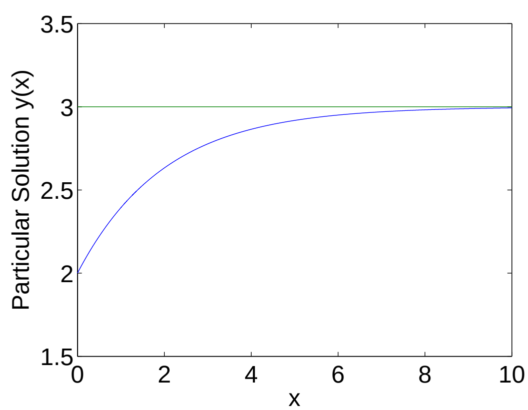


Figure 9: Particular Solution

Example 6 Population Dynamics This is an important area of biology. The simplest equation we can use to describe the dynamics of a population is one that assumes that the birth rate, say α, is proportional to the population size; and that the death rate, say β, is also proportional to the population size. If we set $r = \alpha - \beta$, we get the following equation to model population dynamics

$$\frac{dy}{dt} = ry. \tag{12}$$

This is a separable equation! Hence we can solve it using the separation of variables technique: set $M(x) = r$ and $N(y) = y$. Then we have

$$\begin{aligned}\int \frac{dy}{y} &= \int r dt + c\\ \log|y| &= rt + c \text{ we take the exponential and get}\\ |y| &= e^c e^{rt}\end{aligned}$$

If we define c_0 in a way similar to that of example 5, we have that the population dynamics' general solution is

$$y = c_0 e^{rt}.$$

Now, if we know that the population at time $t = 0$ is, say, y_0, we have that the solution can be written as

$$y(t) = y_0 e^{rt}. \tag{13}$$

We consider now two separate cases.

1. Assume $r < 0$, or $\beta > \alpha$. This means that the death rate is higher than the birth rate. We see that if t is very large, $y_0 e^{rt} \approx 0$. Thus we predict successfully that the population will go extinct.
2. Assume now $r > 0$, or $\beta < \alpha$. This means that the birth rate is higher than the death rate. We expect the population size to increase, but our model predicts that the population will grow *exponentially fast* and that there is no bound to the population size.

The second prediction from our model is clearly wrong. As the population size gets larger, we can still assume that the birth rate is proportional to the population size, but scarcity of resources will make the death rate higher as the population increases.
This motivates our second model, which is called the *logistics equation*. We replace r by $r - ay$, i.e. the death rate now increases linearly with the population size. Thus we have that the equation 12 becomes

$$\frac{dy}{dt} = (r - ay)y = r\left(1 - \frac{y}{r/a}\right)y.$$

Define $K = r/a$, and call it the *carrying capacity*. We also call r the *intrinsic growth rate*. Thus the equation becomes what is known as the *Logistics Equation*:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y. \tag{14}$$

This is a first order non-linear ODE. Luckily, it is separable. If we set $M(x) = r$ and $N(y) = \left(1 - \frac{y}{K}\right)y$, we get

$$\begin{aligned}\int \frac{dy}{\left(1 - \frac{y}{K}\right)y} &= \int r dt + c\\ \int \frac{dy}{\left(1 - \frac{y}{K}\right)y} &= rt + c\end{aligned}$$

To integrate the left hand side, we use *partial fractions*. We refer you to mathematics textbooks for a complete treatment of partial fractions. The idea is the find A and B such that

$$\frac{1}{\left(1-\frac{y}{K}\right)y} = \frac{A}{1-\frac{y}{K}} + \frac{B}{y}$$

If we multiply this equation by $\left(1-\frac{y}{K}\right)y$, we get

$$1 = Ay + B\left(1-\frac{y}{K}\right)$$

or, by rearranging the previous equation,

$$1 = B + y\left(A - \frac{B}{K}\right).$$

We can view this as an equality between two polynomials in y. Recall that two polynomials are equal if and only if all of its coefficients are equal. Thus we have that, from the constant coefficient

$$B = 1$$

and, from the linear coefficient,

$$0 = A - \frac{B}{K} \text{ or } A = \frac{1}{K}.$$

We conclude that

$$\begin{aligned}\frac{1}{\left(1-\frac{y}{K}\right)y} &= \frac{1/K}{1-\frac{y}{K}} + \frac{1}{y}\\ &= \frac{1}{K-y} + \frac{1}{y}\end{aligned}$$

and now both terms can be easily integrated

$$\begin{aligned}\int \frac{dy}{\left(1-\frac{y}{K}\right)y} &= \int \frac{1}{K-y} + \int \frac{1}{y}\\ &= -\log|K-y| + \log|y|\\ &= \log\frac{|y|}{|K-y|}\end{aligned}$$

where the last equality holds because $\log(a) - \log(b) = \log(a/b)$.
Recall that

$$\int \frac{dy}{\left(1-\frac{y}{K}\right)y} = rt + c$$

hence we conclude that

$$\begin{aligned}
\log \frac{|y|}{|K-y|} &= rt + c, \qquad \text{we take the exponential and get} \\
\frac{|y|}{|K-y|} &= \exp(rt+c) \\
\frac{|y|}{|K-y|} &= e^c e^{rt}, \qquad \text{and defining } c_0 \text{ as before} \\
\frac{y}{K-y} &= c_0 e^{rt}
\end{aligned}$$

We multiply both sides by $K-y$ to get

$$\begin{aligned}
y &= c_0 e^{rt}(K-y), \qquad \text{solve for } y \\
y\left(1 + c_0 e^{rt}\right) &= K c_0 e^{rt}, \qquad \text{thus we get} \\
y(t) &= \frac{K c_0 \exp(rt)}{1 + c_0 \exp(rt)}, \qquad \text{which can be written as} \\
y(t) &= \frac{K}{\frac{1}{c_0 \exp(rt)} + 1}
\end{aligned}$$

and finally, we get

$$y(t) = \frac{K}{\frac{\exp(-rt)}{c_0} + 1}. \tag{15}$$

We can now see why this is a better model for population evolution. As t goes to infinity, the ratio $\exp(-rt)/c_0$ converges to zero. We conclude that, as t goes to infinity, the population tends to K. Thus all non-zero solutions tend to K as t goes to infinity.

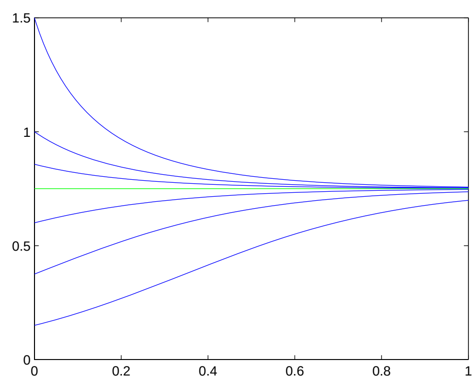


Figure 10: Solutions to Logistics Equation

Let us do a more detailed analysis of the solution and look for equilibrium points. We need to solve for the zeros of the following polynomial:

$$f(y) = r\left(1 - \frac{y}{K}\right) y.$$

It is easy to see that $y = K$ and $y = 0$ are the only zeros of that polynomial. We conclude that the only equilibrium points are 0 and K.
These two equilibrium points have a very natural interpretation. First let us consider $y = 0$. This equilibrium point simply says that if we do not have any specimen, we cannot have any births or deaths. Now consider a small positive perturbation to $y = 0$. We then have that $y' = r(1 - y/K)y \sim ry > 0$, which implies that the population is augmenting very quickly. Thus $y = 0$ is an unstable equilibrium point.
Let us now consider $y = K$. In that case:

$$\begin{aligned} f(y) > 0, y \in (0, K) \\ f(y) < 0, y \in (K, +\infty) \end{aligned}$$

Thus we conclude that a positive perturbation to $y = K$ leads to a negative value for $f(y)$, thus to a negative value for y'. Similarly, a negative perturbation to $y = K$ leads to a positive value for y'. We conclude that $y = K$ must be a stable equilibrium.
A more elaborate model might consider death rates coming from two sources:

- limited resources
- predators

Such model can produce more complex behavior for the evolution of the population.

5 Exact Equation Technique

For the "Exact Equation" technique, we need to consider x and y as two independent variables. Consider f a function of x and y. Its total differential (also simply called differential) is defined as:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{16}$$

The interpretation of the total differential of f at (x, y) is the following. If we calculate f at $(x + dx, y + dy)$ for some small dx and dy, the change in f is df.
Now assume that $f(x, y) = c$ for some constant c. Then we have that $df = 0$, which translates into

$$\begin{aligned} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0, \quad \text{if we divide both sides by } dx: \\ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \end{aligned}$$

and this is an ODE! Thus the solutions to the ODE

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' = 0 \tag{17}$$

are given by $f(x, y) = c$ for c constant.
There is a caveat for this technique. Assume that we have an ODE as in Equation 17. We then know that the solutions satisfy $f(x, y) = c$, but this does not guarantee that we can calculate y as a function of x **explicitly.**
Example 7 Consider the equation $f(x, y) = x^2 + y^2 = 1$. The total differential of f here is given by

$$df = 2xdx + 2ydy$$

f being a constant implies that $df = 0$, thus we have that $2xdx + 2ydy = 0$, which leads to the following ODE

$$2x + 2y\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

We can give the solutions to such equation:

$$x^2 + y^2 = c, \quad \text{or} \quad y = \pm\sqrt{c - x^2}$$

Solution method. Let us now concentrate on the solution method. Assume that you are given a differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0. \tag{18}$$

It is important to realize that this is just a different way to write differential equations. From Equation 18 for example, we get

$$y' = -\frac{M(x, y)}{N(x, y)}$$

We say that a differential equation is *exact* if there exists a function $f(x, y)$ such that

$$df = M(x, y)dx + N(x, y)dy$$

Such a function f must therefore satisfy:

$$\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y).$$

If a differential equation is exact, we know that its solutions are given by the equation

$$f(x, y) = c$$

with c constant.

Proposition 1 (Key Result)**.** *A necessary and sufficient condition for the equation to be exact is that:*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example 8 Assume we are given the following differential equation $2xydx + (x^2-1)dy = 0$. Thus we have that $M(x,y) = 2xy$ and $N(x,y) = (x^2-1)$. From this we have

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x.$$

We can then conclude that this equation is exact. We now have to find the function f. In order to do so, we start with the equation $\frac{\partial f}{\partial x} = M(x,y) = 2xy$ and integrate it with respect to x. Thus, in this operation, y is a constant. Also, in lieu of a constant of integration, we have a function of y, say $g(y)$. After integration we obtain:

$$f(x,y) = x^2y + g(y). \tag{19}$$

To find the function $g(y)$, we use the second equation, namely $\frac{\partial f}{\partial y} = N(x,y)$. We differentiate Equation 19 with respect to y:

$$\frac{\partial f}{\partial y} = x^2 + g'(y).$$

From which, by setting this equal to $N(x,y)$, we obtain the equation:

$$x^2 + g'(y) = x^2 - 1, \quad \text{or} \quad g' = -1.$$

At this point, it is important to verify that this equation does not depend on x. Indeed, g is a function of y only, so if the variable x is still present, a mistake has been made!
We can now solve for g: $g(y) = -y$. We do not include an integration constant as we will later set f to be equal to a constant. Thus the solutions to the initial differential equation are solutions to

$$f(x,y) = c \quad \text{or} \quad x^2y - y = c$$

We can solve this equation directly and get

$$y(x) = \frac{c}{x^2-1}.$$

Now, not all equations are exact. When they are not, the method of integrating factors can make them exact. In the following subsection we explore such technique.

5.1 Integrating Factors Technique

The method of integrating factors for exact equations is the most complicated one we will consider in this course. This is also a very important method as we can use it to reduce any first order **linear** ODE into an exact equation.
Let us assume that $M(x,y)dx + N(x,y)dy = 0$ is not exact. How can we make it exact? It is always possible to multiply the equation by an arbitrary function; this may help transform

the equation into an exact one. We introduce a new function, $\mu(x, y)$, and multiply both sides of the equation by μ:

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0.$$

We then look for μ such that this new equation is exact. In other words, we want to find μ such that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

which leads to the following PDE

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}.$$

But this looks very bad! We started with an ODE and we are now trying to solve a PDE. We thus make a key simplification: *μ is a function of x only or y only.* Of course this is not always true. But if such a μ exists, we now should be able to find it.
We will do the analysis when μ is just a function of x, the case where μ is just a function of y can be done in a similar way.
Given that μ is just a function of x, we have that $\frac{\partial \mu}{\partial y} = 0$, which leads to the following ODE (where the unknown is μ):

$$\mu\frac{\partial M}{\partial y} = \frac{d\mu}{dx}N + \mu\frac{\partial N}{\partial x}$$

which can be rewritten as

$$\frac{d\mu}{dx}N = \mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x}$$
$$\frac{\frac{d\mu}{dx}}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

This only works if the right hand side is a function of x only because this was our assumption. So:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

must be a function of x only. If this is the case, then we can solve for μ by integrating the following equation

$$\frac{\frac{d\mu}{dx}}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

Once we have the solution μ, we have an exact equation of the form:

$$M_{\text{exact}}(x, y)dx + N_{\text{exact}}(x, y)dy = 0$$

where

$$M_{\text{exact}}(x, y) = \mu(x)M(x, y)$$
$$N_{\text{exact}}(x, y) = \mu(x)N(x, y)$$

N.B.: for the case where we consider μ as a function of y only, the following expression must be a function of y only:

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

If this is the case, then we can solve for μ by integrating the following equation

$$\frac{\frac{d\mu}{dy}}{\mu} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

This time:

$$M_{\text{exact}}(x, y) = \mu(y) M(x, y)$$
$$N_{\text{exact}}(x, y) = \mu(y) N(x, y)$$

Example 9 Assume we have the following ODE to solve

$$xydx + (2x^2 + 3y^2 - 20)dy = 0. \tag{20}$$

Let's see if this equation is exact:

$$M(x, y) = xy, \quad \text{thus} \quad \frac{\partial M}{\partial y} = x$$
$$N(x, y) = 2x^2 + 3y^2 - 20, \quad \text{thus} \quad \frac{\partial N}{\partial x} = 4x$$

and we conclude that the equation is not exact. Let us see if we can find an integrating factor μ that is only a function of x. For that we need the expression

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20}$$

to be a function of x only. It does not work.
We now see if we can look for an integrating factor μ that is only a function of y. To that end, we need the expression

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}$$

to be a function of y only. Yes! This is the case.
We know that μ must be a solution to the following ODE

$$\frac{\frac{d\mu}{dy}}{\mu} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{3}{y}$$

which can be integrated to yield

$$\ln|\mu| = 3\ln|y| + c.$$

Recall that, for all a and b we have that $a \ln|b| = \ln\left(|b|^a\right)$, thus we have that

$$\begin{aligned}
\ln|\mu| &= \ln\left(|y|^3\right) + c, \quad \text{we take the exponential and get} \\
|\mu| &= e^c |y|^3, \quad \text{by replacing } e^c \text{ with } c_0, \text{ we get} \\
\mu &= c_0 y^3
\end{aligned}$$

We can drop the constant c_0 as it will only appear as a multiplicative factor in the original Equation 20 after multiplying it by μ. Thus:

$$\mu(y) = y^3$$

is what we need. We multiply Equation 20 by μ to get the following exact equation:

$$xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0. \tag{21}$$

We can indeed verify that this equation is exact.

$$\begin{aligned}
M(x,y) &= xy^4, \quad \text{thus} \quad \frac{\partial M}{\partial y} = 4xy^3 \\
N(x,y) &= 2x^2y^3 + 3y^5 - 20y^3, \quad \text{thus} \quad \frac{\partial N}{\partial x} = 4xy^3
\end{aligned}$$

which proves that Equation 21 is exact.
We now solve this equation using the Exact Equation technique. Recall that we are looking for a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M(x,y), \quad \frac{\partial f}{\partial y} = N(x,y).$$

Thus we have that

$$\frac{\partial f}{\partial x} = xy^4,$$

which we can integrate with respect to x to get

$$f(x,y) = \frac{1}{2}x^2y^4 + g(y). \tag{22}$$

We now differentiate Equation 22 with respect to y, and use $\frac{\partial f}{\partial y} = N(x,y)$ to get

$$2x^2y^3 + \frac{dg}{dy} = N(x,y) = 2x^2y^3 + 3y^5 - 20y^3.$$

We solve for g':

$$\begin{aligned}
\frac{dg}{dy} &= N(x,y) - 2x^2y^3 \\
&= 2x^2y^3 + 3y^5 - 20y^3 - 2x^2y^3 \\
&= 3y^5 - 20y^3,
\end{aligned}$$

which we can directly integrate to get

$$g(y) = \frac{1}{2}y^6 - 5y^4.$$

Note that all the terms containing x are gone! We now conclude that the solutions to the differential equation 21 are given by

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$$

where c is a constant.
Matlab can be used to plot this complicated solution. If we assume that $c = 10$, we only need to enter the command

```
ezplot('1/2*x^2*y^4 + 1/2*y^6 - 5*y^4=10')
```

and we get Figure 11. Since y appears only with even powers, y and $-y$ are solutions of this equation as evidenced by the figure.

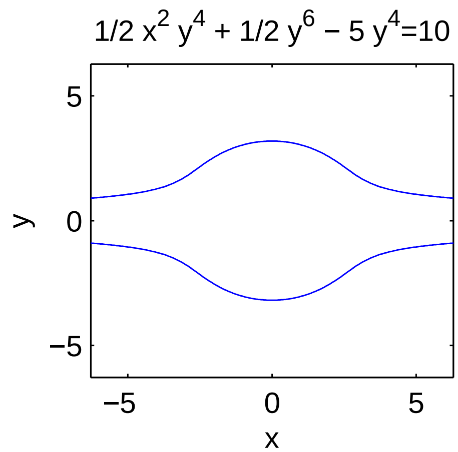


Figure 11: Plot of the Solution of Equation 21

6 Linear Differential Equations Technique

We first present this technique from the view point of exact equations and integrating factors. Once we have established the general form of the solution we will present a shorter technique to arrive at the same result. The second approach is easier to memorize.
Consider a linear first order differential equation in its standard form

$$\frac{dy}{dx} + p(x)y(x) = r(x) \tag{23}$$

Remark that a first order ODE of the form $q(x)y' + p(x)y(x) = r(x)$ can be reduced to standard form by dividing it by $q(x)$.

Is equation 23 exact? Let us check. We re-write the equation as:

$$(p(x)y - r(x))\,dx + dy = 0$$

$$\frac{\partial(p(x)y - r(x))}{\partial y} = p(x), \quad \frac{\partial(1)}{\partial x} = 0$$

This equation is not exact. However we are going to see that the technique of Integrating Factors is applicable here. Here $M = p(x)y - r(x)$ and $N = 1$. Let us see if $(\partial M/\partial y - \partial N/\partial x)/N$ is a function of x only:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{p(x) - 0}{1} = p(x)$$

Yes! It is a function of x only. So we can compute $\mu(x)$. It is the solution of:

$$\frac{\mu'}{\mu(x)} = p(x) \quad \therefore\ \mu(x) = \mathrm{e}^{\int p(x)dx}$$

Let us multiply our differential equation by μ and we obtain:

$$\mathrm{e}^{\int p(x)dx}(p(x)y - r(x))\,dx + \mathrm{e}^{\int p(x)dx}\,dy = 0$$

You can check that this equation is exact. The solution can be easily found. We look for a function f which satisfies:

$$\frac{\partial f}{\partial x} = \mathrm{e}^{\int p(x)dx}(p(x)y - r(x)), \quad \frac{\partial f}{\partial y} = \mathrm{e}^{\int p(x)dx}$$

Let us integrate the first equation:

$$f = \mathrm{e}^{\int p(x)dx}\,y - \int \mathrm{e}^{\int p(x)dx}\,r(x)\,dx + g(y)$$

Differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \mathrm{e}^{\int p(x)dx} + g'$$

This expression must be equal to $\mathrm{e}^{\int p(x)dx}$. So g is a constant. The solution y therefore simply satisfies:

$$\mathrm{e}^{\int p(x)dx}\,y - \int \mathrm{e}^{\int p(x)dx}\,r(x)\,dx = c$$

We can solve for y to obtain an explicit solution:

$$\boxed{y = \mathrm{e}^{-\int p(x)dx}\left(\int \mathrm{e}^{\int p(x)dx}\,r(x)\,dx + c\right)} \tag{24}$$

This is a long expression but it is the general solution to our problem. In this class, we allow a sheet of formulas in exams. So now you have at least one formula to put in that sheet of paper.

Here is another derivation which is easier to memorize than the previous one. However you will see that it is not as elegant as the previous approach.
Consider the **homogeneous** case where the right-hand side is 0:

$$\frac{dy}{dx} + p(x)y = 0. \tag{25}$$

It can be solved by the separation of variables method. It yields

$$\frac{dy}{y} = -p(x)dx$$

which can be integrated

$$\begin{aligned} \log|y| &= -\int p(x)dx + c, \quad \text{we take the exponential and get} \\ |y| &= e^c\, \mathrm{e}^{-\int p(x)dx}. \end{aligned}$$

By choosing c_0 the usual way, we get the general solution for a first order linear homogeneous ODE:

$$y(x) = c_0\, e^{-\int p(x)dx}. \tag{26}$$

Let us consider the **in-homogeneous** case with a general right-hand side:

$$\frac{dy}{dx} + p(x)y(x) = r(x).$$

Consider the following idea. Take $y(x)$ a solution and write it in the form:

$$y(x) = u(x)\, e^{-\int p(x)dx}. \tag{27}$$

It is as if we had replaced the constant c_0 in Equation 26 by a function $u(x)$. Note that it is always possible to write $y(x)$ in this form.
We then use the fact that y is a solution to get a new equation for $u(x)$:

$$\begin{aligned} \frac{dy}{dx} + p(x)y &= r(x) \\ \frac{d}{dx}\left[u(x)\mathrm{e}^{-\int p(x)dx}\right] + p(x)u(x)\mathrm{e}^{-\int p(x)dx} &= r(x) \\ \frac{du}{dx}\mathrm{e}^{-\int p(x)dx} + u(x)\frac{d}{dx}\left(\mathrm{e}^{-\int p(x)dx}\right) + p(x)u(x)\mathrm{e}^{-\int p(x)dx} &= r(x) \\ \frac{du}{dx}\mathrm{e}^{-\int p(x)dx} + u(x)(-p(x))\mathrm{e}^{-\int p(x)dx} + p(x)u(x)\mathrm{e}^{-\int p(x)dx} &= r(x) \\ \frac{du}{dx}\mathrm{e}^{-\int p(x)dx} &= r(x) \end{aligned}$$

which simplifies to the following equation

$$\frac{du}{dx} = r(x)e^{\int p(x)dx}. \tag{28}$$

And this equation can be solved directly! We integrate and get

$$u(x) = \int r(x)e^{\int p(x)dx}dx + c.$$

From Equation 27, we get the general solution:

$$y = \mathrm{e}^{-\int p(x)dx}\left(\int \mathrm{e}^{\int p(x)dx}\, r(x)\, dx + c\right)$$

Example 10 Consider the following first order linear ODE

$$xy' - 4y = x^6\mathrm{e}^x$$

In order to put it in standard form, we divide by x and get

$$y' - \frac{4}{x}y = x^5\mathrm{e}^x. \tag{29}$$

First solve the homogeneous equation:

$$y' - \frac{4}{x}y = 0 \quad \therefore\ y(x) = c\,x^4$$

Following the method we described previously, we consider $y(x) = u(x)x^4$. Then, inserting this in Equation 29:

$$\begin{aligned}
[u(x)x^4]' - \frac{4}{x}\,u(x)x^4 &= x^5\mathrm{e}^x, \\
u(x)'x^4 + u(x)\,4x^3 - 4\,u(x)x^3 &= x^5\mathrm{e}^x, \\
u'x^4 &= x^5\mathrm{e}^x, \\
u' &= x\mathrm{e}^x, \\
u &= (x-1)\mathrm{e}^x + c
\end{aligned}$$

(Use Matlab or integration by parts for the last line.)
Finally:

$$y(x) = x^4\big((x-1)\mathrm{e}^x + c\big)$$

6.1 Summary

The method for linear first order ODEs concluded our treatment of techniques to solve first order ODEs. Recall that we studied three techniques for solving ODEs:

1. Separable Equations.
   These are equations of the form $M(y)dy = N(x)dx$. We can solve them by integrating directly the ODE.

2. Exact Equations.
   An exact equation of the form $M(x,y)dx + N(x,y)dy = 0$ is such that there exists a function $f(x,y)$ with $\partial f/\partial x = M$, $\partial f/\partial y = N$. The solutions of this equation are given by $f(x,y) = c$ for c constant.

   If an equation is not exact, one can use the method of integrating factors to try to modify the original equation and get an exact one.

3. Linear Equations.
   A linear equation in its standard form is $y' + p(x)y = r(x)$. The analytical solution is given by
$$y(x) = \mathrm{e}^{-\int p(x)dx}\left[\int r(x)\,\mathrm{e}^{\int p(x)dx}dx + c\right] \tag{30}$$

Before we jump into second order equations, let us study *models for electrical circuits*.

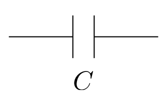
7 Modeling Electrical Circuits

The study of electrical circuits is very important. The equations we will derive are relevant not only to electrical circuits, but to many other fundamental problems including many mechanical systems. We will begin by presenting the different components that we will consider in the electrical circuits we will model.

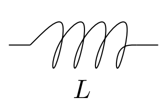
7.1 Electrical Components

We will be using four elementary electrical components.

- Capacitor C. A capacitor is used to store electrical charges. The voltage drop in a capacitor is Q/C, where Q is the charge stored in the capacitor and C is given in farads. The figure below is the usual representation of a capacitor.



- Inductor L. The voltage drop in an inductor is $L\frac{dI}{dt}$. If an inductor is present, the current in the electrical circuit cannot be discontinuous. An inductor is used to store electromagnetic energy in its coil. L is given in henrys. The figure below is the usual representation of an inductor.



- Resistor R. A resistor dissipates electromagnetic energy into heat. The voltage drop in a resistor is equal to (Ohm's law) RI. R is in ohms. The figure below is the usual representation of a resistor.

- Imposed voltage $E(t)$. The figure below is the usual representation of an imposed voltage.

The simplest electrical circuit one can study containing these four components is represented in Figure 12.

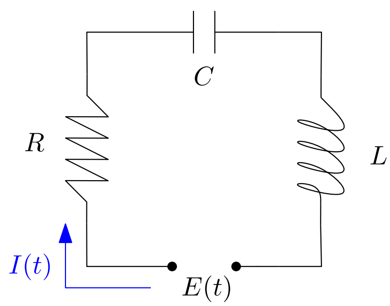


Figure 12: RLC electrical circuit

The question we want to answer is the following. Given an imposed voltage $E(t)$, can we calculate the current intensity in the circuit $I(t)$?

7.2 Kirchhoff Law and Relation Between Q and I

Kirchhoff law allows us to relate the imposed voltage to the sum of the voltage drops for each component. For the RLC circuit from Figure 12 we get

$$E(t) = I(t)R + L\frac{dI}{dt} + \frac{Q}{C} \tag{31}$$

But we are not done yet. We still need to link Q to $I(t)$. Recall that Q is the amount of electric charges stored in the capacitor. By conservation of electrons, we have that

$$\frac{dQ}{dt} = I(t) \tag{32}$$

We now need to solve *simultaneously* Equations 31 and 32. And this is a problem as we have a system of differential equations instead of just an ODE! In order to get an ODE, we use a simple trick: we differentiate with respect to t Equation 31 and get

$$\frac{dE}{dt} = \frac{dI}{dt}R + L\frac{d^2I}{dt^2} + \frac{1}{C}\frac{dQ}{dt}$$

we then use Equation 32 to finally get

$$\frac{dE}{dt} = L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I(t). \tag{33}$$

This is a second order, linear, inhomogeneous ODE. In addition, all the coefficients are constant. If the coefficients are not constant, it is very difficult to solve such equations.

7.3 RC Electrical Circuit

In this section, we assume that the voltage drop due to the inductor is negligible, thus we set $L = 0$. Figure 13 represents the electrical circuit.

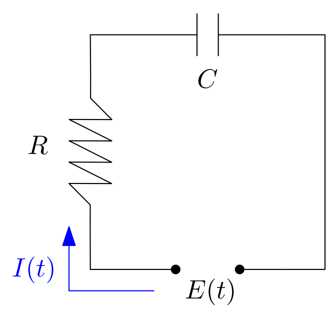


Figure 13: RC electrical circuit

From Equation 33, after setting $L = 0$, we get

$$\frac{dE}{dt} = R\frac{dI}{dt} + \frac{1}{C}I(t)$$

which is now a first order equation. We still need to put it into standard form. We divide by R to get

$$\frac{dI}{dt} + \frac{1}{RC}I(t) = \frac{1}{R}\frac{dE}{dt} \tag{34}$$

Let $t_C = RC$. t_C is the capacitance time. This time is related to the time it takes a capacitor to charge. We will see later why this is true.
Let us solve this equation. The homogeneous equation has the following solution:

$$I' + \frac{1}{t_C}I(t) = 0, \quad \therefore\ I = I_0\,e^{-t/t_C}.$$

We consider solutions of the form $I = u(t)\,\mathrm{e}^{-t/t_\mathrm{C}}$. Then:

$$\begin{aligned} I' + \frac{1}{t_\mathrm{C}} I(t) &= \frac{E'}{R} \\ [u(t)\,\mathrm{e}^{-t/t_\mathrm{C}}]' + \frac{1}{t_\mathrm{C}}\,u(t)\,\mathrm{e}^{-t/t_\mathrm{C}} &= \frac{E'}{R} \\ u(t)'\,\mathrm{e}^{-t/t_\mathrm{C}} - \frac{1}{t_\mathrm{C}} u(t)\,\mathrm{e}^{-t/t_\mathrm{C}} + \frac{1}{t_\mathrm{C}}\,u(t)\,\mathrm{e}^{-t/t_\mathrm{C}} &= \frac{E'}{R} \\ u(t)'\,\mathrm{e}^{-t/t_\mathrm{C}} &= \frac{E'}{R} \\ u(t) &= \int \frac{E'}{R}\,\mathrm{e}^{t/t_\mathrm{C}}\,dt + c \end{aligned}$$

We get the general solution for the current intensity given the imposed voltage:

$$I(t) = \mathrm{e}^{-\frac{t}{t_\mathrm{C}}} \left[\int \frac{E'}{R}\,\mathrm{e}^{\frac{t}{t_\mathrm{C}}}\,dt + c \right] \tag{35}$$

In order to simplify this formula, we are going to consider two important cases for the imposed voltage $E(t)$.

Battery. A constant voltage $E(t) = E_0$ in used.
In this case, we have that $\frac{dE}{dt} = 0$, thus Equation 35 simplifies to

$$I(t) = c\,\mathrm{e}^{-\frac{t}{t_\mathrm{C}}}. \tag{36}$$

In Figure 14 we plot the current intensity for $I_0 = 10$ and $t_\mathrm{C} = 1$. As you can see, after just $1\,t_\mathrm{C}$, the value of I is less than 40% that of I_0, and after $2\,t_\mathrm{C}$ it is less than 20%. Therefore, we can say that $I(t) \approx 0$ when $t \gg t_\mathrm{C}$.

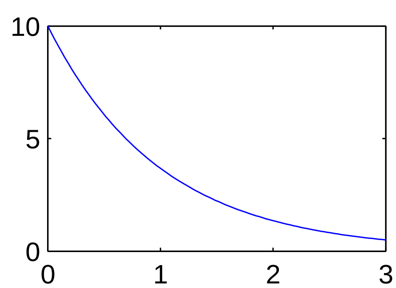


Figure 14: Current intensity in RC circuit with $E(t) = E_0$

But typically $E(t)$ is not constant, so we consider next one of the most common functions for the imposed voltage.

Sine voltage. $E(t) = E_0 \sin(\omega t)$. This is the type of voltage we get from the outlet. $\frac{\omega}{2\pi}$ is the frequency in hertz (in the USA 60 hertz, in Europe usually 50 hertz). Thus the period of the voltage is $T = \frac{2\pi}{\omega}$.

Here we have that

$$\frac{1}{R}\frac{dE}{dt} = \frac{1}{R}E_0\omega\cos(\omega t).$$

Thus, in order to calculate the solution to the ODE, we need to calculate

$$\int \frac{1}{R}E_0\omega\cos(\omega t)\,\mathrm{e}^{\frac{t}{t_\mathrm{C}}}\,dt = \frac{E_0\omega}{R}\int \cos(\omega t)\,\mathrm{e}^{\frac{t}{t_\mathrm{C}}}\,dt. \tag{37}$$

To calculate $\int\cos(\omega t)\mathrm{e}^{\frac{t}{t_\mathrm{C}}}dt$, one needs to apply the integration by parts formula twice. Look up your course textbook on the first page or use Matlab to find that:

$$\int \cos(\omega t)e^{\frac{t}{t_\mathrm{C}}}dt = \frac{\omega}{\frac{1}{t_\mathrm{C}^2}+\omega^2}\,\mathrm{e}^{\frac{t}{t_\mathrm{C}}}\left(\frac{1}{\omega t_\mathrm{C}}\cos(\omega t)+\sin(\omega t)\right)$$

In Figure 15 we plot $\left(\frac{1}{\omega t_\mathrm{C}}\cos(\omega t)+\sin(\omega t)\right)$. This function is simply a shifted sine function with a certain amplitude. Is it possible to re-write our answer so that we can find explicitly the shift and the amplitude of the oscillations?

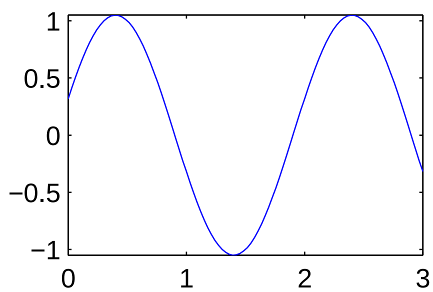


Figure 15: Shifted sine function

In order to get $\frac{1}{\omega t_\mathrm{C}}\cos(\omega t)+\sin(\omega t)$ factored into a shifted sine function, we use the following

$$\begin{aligned}\sin(\omega t+\delta) &= \sin(\omega t)\cos(\delta)+\cos(\omega t)\sin(\delta)\\ &= \cos(\delta)\left[\sin(\omega t)+\cos(\omega t)\frac{\sin(\delta)}{\cos(\delta)}\right]\\ &= \cos(\delta)\left[\tan(\delta)\cos(\omega t)+\sin(\omega t)\right]\end{aligned}$$

Therefore we choose δ such that $\tan\delta = 1/(\omega t_\mathrm{C})$; δ is the shift we were looking for. To find $\cos(\delta)$, consider Figure 16. From the geometrical definition of the cosine function we have:

$$\cos(\delta) = \frac{\omega t_\mathrm{C}}{\sqrt{1+\omega^2 t_\mathrm{C}^2}}$$

With this value of δ

$$\sin(\omega t+\delta) = \frac{\omega t_\mathrm{C}}{\sqrt{1+\omega^2 t_\mathrm{C}^2}}\left(\frac{1}{\omega t_\mathrm{C}}\cos(\omega t)+\sin(\omega t)\right).$$

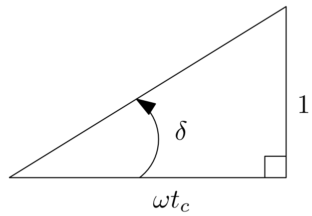


Figure 16: Useful triangle for trigonometric identities

Recall that our equation is:

$$\int \frac{E_0\omega}{R}\cos(\omega t)\,\mathrm{e}^{\frac{t}{t_C}}\,dt = \frac{E_0\omega}{R}\frac{\omega\mathrm{e}^{\frac{t}{t_C}}}{\frac{1}{t_C^2}+\omega^2}\left(\frac{1}{\omega t_C}\cos(\omega t)+\sin(\omega t)\right)$$

We combine both equations and get:

$$\begin{aligned}\int \frac{E_0\omega}{R}\cos(\omega t)\,\mathrm{e}^{\frac{t}{t_C}}\,dt &= \frac{E_0\omega}{R}\frac{\omega t_C^2\,e^{\frac{t}{t_C}}}{1+\omega^2 t_C^2}\left(\frac{1}{\omega t_C}\cos(\omega t)+\sin(\omega t)\right)\\ &= \frac{E_0\omega}{R}\mathrm{e}^{\frac{t}{t_C}}\frac{t_C}{\sqrt{1+\omega^2 t_C^2}}\sin(\omega t+\delta)\end{aligned}$$

We now use Equation 35 to get

$$I(t) = \mathrm{e}^{-\frac{t}{t_C}}\left[\frac{E_0\omega t_C}{R\sqrt{1+\omega^2 t_C^2}}\mathrm{e}^{\frac{t}{t_C}}\sin(\omega t+\delta)+c\right],$$

which can be rewritten as

$$I(t) = c\mathrm{e}^{-\frac{t}{t_C}} + \frac{E_0\omega t_C}{R\sqrt{1+\omega^2 t_C^2}}\sin(\omega t+\delta) \tag{38}$$

Recall that δ is such that $\tan(\delta) = \frac{1}{\omega t_C}$ and $t_C = RC$.
We can define the **impedance** of the circuit as

$$R_0 = \frac{R\sqrt{1+\omega^2 t_C^2}}{\omega t_C} = \sqrt{R^2+\frac{R^2}{\omega^2 t_C^2}}$$

Since $t_C = RC$, the impedance is also given by:

$$R_0 = \sqrt{R^2+\frac{1}{\omega^2 C^2}}.$$

The impedance is expressed in ohms, just like the resistance. With this notation, we finally get this simple expression for the current when the imposed voltage is a sine function:

$$\boxed{I(t) = c\,\mathrm{e}^{-\frac{t}{t_C}} + \frac{E_0}{R_0}\sin(\omega t+\delta)}$$

We have two very different terms in this equation:

1. A *transient* response: $c\,\mathrm{e}^{-\frac{t}{t_\mathrm{C}}}$. This term is called transient as it will play no role as $t \gg t_\mathrm{C}$. This term is also identical to the current intensity when we are charging the circuit with a battery. The value of c depends on the initial condition.

2. A *steady state* response: $(E_0/R_0)\sin(\omega t + \delta)$. Be careful not to confuse steady state response and steady state solution (= a constant solution)! This is the solution observed once $t \gg t_\mathrm{C}$.

We can see in Figure 17 a plot of the general solution to the *RC* circuit's current intensity. The transient response does not last more than $3\,t_\mathrm{C}$ (here $t_\mathrm{C} = 1$).

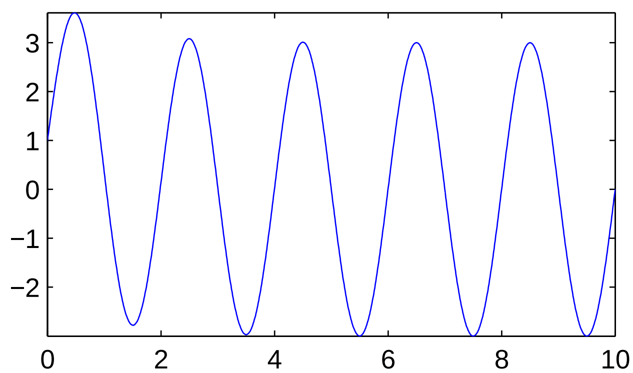


Figure 17: Current intensity in a *RC* circuit

8 Symbolic Solutions to ODEs using Matlab

Matlab has a symbolic toolbox which can solve certain ODEs analytically. In this section we investigate some useful commands from the symbolic toolbox. In order to reduce the number of empty lines in Matlab's output, we use the command

```
>> format compact
```

- We start with differentiation. The differentiation command is `>> diff('')`. Here are some examples of the use of such command

```
>> diff('cos(x)')
ans =
-sin(x)
>> diff('atan(x)')
ans =
1/(1+x^2)
```

- We can also integrate. The integration command is `>> int('')`. Here are some examples of the use of such command

  ```
  >> int('1/(1-x^2)')
  ans =
  atanh(x)
  >> int('1/((x-1)*(x-2))')
  ans =
  -log(x-1)+log(x-2)
  ```

  As we can see from the last example, it even does partial fractions. But in order to solve ODEs we need more than this.

- We use the command `>> dsolve` to solve ODEs with Matlab symbolically. As this command is more complex than the previous ones, it is recommended to read what Matlab's help says about it.

  Important Remark: when using the command `dsolve`, the default independent variable is `t`. This is a source of many errors and cannot be overstressed. It is usually safer to always use `t`. If you want to use another variable, like `x`, you should add `'x'` at the end of the `dsolve` input arguments. You will see an example later on.

Let us now focus on some examples of the use of `dsolve` where we already know what the analytical solution is.

Example 11 Coffee Temperature

Recall that the ODE for coffee's temperature is given by

$$\frac{dT}{dt} = -k(T - T_a).$$

In order to input this equation, we need to learn `dsolve`'s syntax to represent derivatives. The following summarizes how to input derivatives into `dsolve`:

$$\frac{df}{dt} \to Df$$
$$\frac{d^2 f}{dt^2} \to D2f$$
$$\frac{d^3 f}{dt^3} \to D3f$$
$$\frac{d^4 f}{dt^4} \to D4f$$

Thus we can now solve our ODE

```
>> dsolve('DT = -k*(T-Ta)')
ans =
Ta+exp(-k*t)*C1
```

which is the solution we found in previous lectures.
We can specify initial conditions in two different ways. Below we specify $T(0) = 100$.

```
>> dsolve('DT = -k*(T-Ta)','T(0)=100')
ans =
Ta+exp(-k*t)*(-Ta+100)
```

or we can even specify a generic initial condition such as $T(0) = T_0$.

```
>> dsolve('DT = -k*(T-Ta)','T(0)=T0')
ans =
Ta+exp(-k*t)*(-Ta+T0)
```

Example 12 Population Dynamics
Recall that the ODE for population dynamics is given by

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P.$$

Let us first solve with no initial condition

```
>> dsolve('DP = r*(1-P/K)*P')
ans =
K/(1+exp(-r*t)*C1*K)
```

We now solve with the generic initial condition $P(0) = P_0$

```
>> dsolve('DP = r*(1-P/K)*P','P(0)=P0')
ans =
K/(1+exp(-r*t)*(K-P0)/P0)
```

Example 13 Pendulum
Recall that the ODE describing the angle of the pendulum, for small values of the angle, is given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

We choose l such that $g/l = 4$, and solve for the resulting equation

```
>> dsolve('D2theta + 4*theta = 0')
ans =
C1*sin(2*t)+C2*cos(2*t)
```

we get a general solution with two arbitrary constants! We'll see in one of the next lectures that this is always the case for second order differential equations. We thus need *two initial conditions*, which are usually given as $\theta(0) = \theta_0$ and $\theta'(0) = \theta'_0$.
But there is something else of interest in this equation. If we let $g/l = \omega^2$, we get

```
>> dsolve('D2theta + w^2*theta = 0')
ans =
C1*sin(w*t)+C2*cos(w*t)
```

Thus we see that

$$\frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

is the frequency of the pendulum.
If we specify only one initial condition, say $\theta(0) = \pi/4$ we get

```
>> dsolve('D2theta + w^2*theta = 0','theta(0)=pi/4')
ans =
C1*sin(w*t)+1/4*pi*cos(w*t)
```

where we still have one integration constant.
Assume now that, initially, the pendulum's velocity was zero. This translates into $\theta'(0) = 0$. We add this initial condition and get

```
>> dsolve('D2theta + w^2*theta = 0','theta(0)=pi/4','Dtheta(0)=0')
ans =
1/4*pi*cos(w*t)
```

and we no longer have any integration constant left.
Example 14 Independent Variable
Assume we want to solve the following ODE

$$x\frac{dg}{dx} - 4g = x^6 e^x.$$

If we are not careful, we would type

```
>> dsolve('x*Dg-4*g=x^6*exp(x)')
ans =
(-1/4*x^6*exp((x^2-4*t)/x)+C1)*exp(4/x*t)
```

but this expression depends on `t`! Here we forgot that Matlab's default independent variable is `t` and not `x`. We can correct this by doing

```
>> dsolve('x*Dg-4*g=x^6*exp(x)','x')
ans =
(x*exp(x)-exp(x)+C1)*x^4
```

which is the solution we were looking for.
Here are two useful remarks about the symbolic toolbox from Matlab

- Beware! when using Matlab's symbolic toolbox you should remember that Matlab does not provide "pretty" solutions, it just provides *a* solution. Thus what Matlab outputs and what you calculate might not resemble each other; however the two answers should be equal. Matlab expressions tend to be quite long.
- Matlab symbolic toolbox comes from Maple. So there is no value added to use Maple instead of the Matlab symbolic toolbox as it is the same set of libraries.

9 Numerical Solutions to ODEs: the Euler method

9.1 Euler method

The goal of this subsection is to understand basic numerical methods to solve ODEs. The methods we will consider to begin with will be quite simple. As we get near the end, we will study Runge-Kutta 4 which is the building block of Matlab's `ode45` command.
The problem we are going to consider is the following

$$\frac{dy}{dt} = f(t, y) \tag{39}$$
$$y(0) = y_0 \tag{40}$$

Bellow are some advantages and disadvantages to solving numerically such ODE.

- Advantages: we can solve *any* ODE.
- Disadvantages:
  - There are errors. We only get an approximation to the solution.
  - Computational cost. The running time can vary from less than a second to weeks, even months. Sometimes it is impossible (because of the required running time) to solve an ODE numerically.
    There are two main fixes to the computational cost problem.
    * Use better numerics.
    * Use more computational power. This can range from getting faster computers to using several computers in parallel.

We begin with the simplest numerical method: the Euler method.
We need to start with a given initial condition and an interval to calculate the solution, say $[a, b]$. We want to compute the solution at n different values $t_1, ..., t_n = 1$:

$$a = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = b.$$

Let the values we compute be $y_0, y_1, ..., y_n$ where y_i is an approximation to the solution at time t_i. Recall that y_0 is exact as it is given by the initial condition.

If we decide to choose a constant *time step* h then the values for t_i will be equally spaced according to:

$$t_1 = t_0 + h$$
$$t_2 = t_0 + 2h$$
$$t_3 = t_0 + 3h$$
$$\dots$$

Call $y(t)$ the exact solution to the ODE. In order to get an approximation to $y(t_1) = y(h)$, we can use a *Taylor Expansion* of y. Recall the Taylor expansion of y:

$$y(h) = y(0) + hy'(0) + \frac{h^2}{2}y''(0) + \ldots$$

where we now assume that $t_0 = 0$. We know that $y(0) = y_0$ by the initial condition 40 and, by Equation 39, $y'(0) = f(0, y(0)) = f(0, y_0)$. Thus we know $y'(0)$ exactly!

Euler Method: we approximate $y(h)$ by $y(0) + hf(t_0, y_0)$.

Figure 18 represents graphically what the Euler method does.

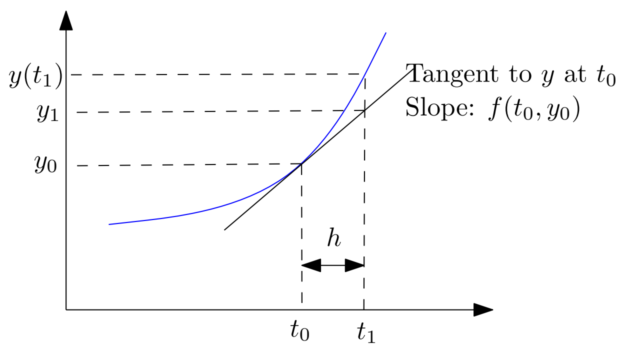


Figure 18: Approximation of $y(t_1)$ using Euler Method

We can iterate these several steps to get

$$y_1 = y_0 + hf(t_0, y_0)$$
$$y_2 = y_1 + hf(t_1, y_1)$$
$$y_3 = y_2 + hf(t_2, y_2)$$
$$y_4 = y_3 + hf(t_3, y_3)$$

It is important to remember that arrays in Matlab, as in any programming language, are indexed by *integers*. Thus Matlab will not understand a command of the form

```
>> y(t1) = y(t0) +  h*f(t0,y(t0));
```

Another important thing to remember is that Matlab's arrays begin at 1 and not at 0. Thus we cannot use $y(0)$ to access the first element in the array. We will need to shift our indices when working with Matlab arrays.

Recall that we are studying the Euler method for the following initial value problem

$$\frac{dy}{dt} = f(t, y)$$
$$y(0) = y_0$$

We are given the solution at $t_0 = 0$ by Equation 40. We will compute the solution in an interval, say $[0, 1]$. In order to do so, we want to compute the solution at n different values $t_1, ..., t_n$ from the initial condition. We further impose that

$$0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = 1.$$

Let the values we compute be $y_0, y_1, ..., y_n$ where y_i is an approximation to the solution at time t_i. Recall that y_0 is exact as it is given by the initial condition 40.
We define the *time step* h as follows.

$$\begin{aligned} t_0 &= 0 \\ t_1 &= h \\ t_2 &= 2h \\ t_3 &= 3h \\ &\ldots\ldots \\ t_n &= nh = 1 \end{aligned}$$

9.2 Matlab implementation of the Euler Method

Recall that in the Euler method, we replace the value of y at t_i with the only two terms of y's Taylor expansion. We thus get

$$y_{i+1} = y_i + hf(t_i, y_i)$$

where $y_{i+1} \approx y(t_{i+1})$.
We are now going to present a Matlab implementation that calculates the first three terms of Euler's method for the following initial value problem

$$\frac{dy}{dt} = 1 - t + 4y$$
$$y(0) = 1.$$

```
function demo_euler
y(1) = 1; t(1) = 0;
h = 0.05;
y(2) = y(1) + h*f(t(1),y(1)); t(2) = h;
y(3) = y(2) + h*f(t(2),y(2)); t(3) = 2*h;
```

```
06 y(4) = y(3) + h*f(t(3),y(3)); t(4) = 3*h;

07 'Comparison of Euler and the exact solution'
08 fprintf('t=0:    %12.5f, \t %12.5f,\n',y(1),exact(0))
09 fprintf('t=0.05: %12.5f, \t %12.5f,\n',y(2),exact(h))
10 fprintf('t=0.1:  %12.5f, \t %12.5f,\n',y(3),exact(2*h))
11 fprintf('t=0.15: %12.5f, \t %12.5f.\n',y(4),exact(3*h))

12 te = linspace(0,3*h,100);
13 plot(te,exact(te),t,y,'o')
14 legend('Exact','Euler')
15 end

16 function y = exact(t)
17 y = 1/4*t-3/16+19/16*exp(4*t);
18 end

19 function yp = f(t,y)
20 yp = 1-t+4*y;
21 end
```

About this implementation:

- As we stated in the previous lecture, indices for arrays in Matlab are positive integers.
- The use of the command `fprintf`. This command is used to display text in the command window. Its syntax is similar to that of the function `fprint` from languages such as C or C++. This command is not essential as one can get Matlab to display on the command window by simply omitting the ";" at the end of a command.
- Finally, the most important feature is the use of a stand-alone function `f` (lines 19 to 21). This allows us to separate the specific problem we are solving from the code used to produce the numerical solution. This is very important and can be related to the notion of "modular code" from software engineering. Keeping the definition of the ODE and the numerical solver separate allows us to improve/change/update any of them independently.

As we can see in Figure 19, the error in the numerical solution seems to be increasing as we move away from the initial condition. It is very important to remember that most of the time only *relative* errors are important.
There are several ways to quantify the error produced by a numerical method:

1. local errors: error incurred at each time-step.

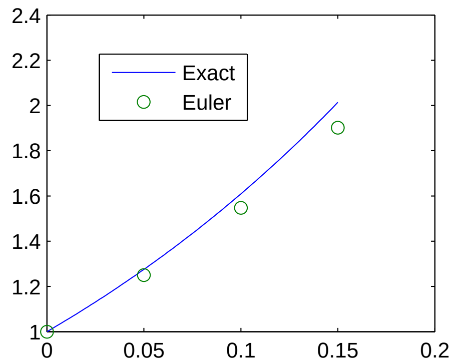


Figure 19: First three steps for Euler method with $h = 0.05$

2. global error: a measure of the error incurred throughout the whole interval where we apply the numerical method.

The second notion is the most important one in engineering. We defer the treatment of error to later in the course.

Now, when we see the previous implementation we notice that such an approach cannot be used in a general setting. Imagine that we need to produce an approximation to the solution at 1000 different points. We do not want to produce 1000 lines with essentially the same code. In order to address this issue, we present a second and final implementation of the Euler method.

```
function demo_euler
y(1) = 1; t(1) = 0;
h = 0.00625;
n=1;
while t(n) < 1
    y(n+1) = y(n) + h*f(t(n),y(n));
    t(n+1) = t(n) + h;
    n = n+1;
end

te = linspace(0,1,100);
plot(te,exact(te),t,y,'+')
legend('Exact','Euler')

abs(y(n)-exact(t(n)))
end

function y = exact(t)
y = 1/4*t-3/16+19/16*exp(4*t);
```

```
17 end

18 function yp = f(t,y)
19 yp = 1-t+4*y;
20 end
```

As we can see, other than removing the display of the result to the command window, we replaced the lines

```
04 y(2) = y(1) + h*f(t(1),y(1)); t(2) = h;
05 y(3) = y(2) + h*f(t(2),y(2)); t(3) = 2*h;
06 y(4) = y(3) + h*f(t(3),y(3)); t(4) = 3*h;
```

with a while loop

```
04 n=1;
05 while t(n) < 1
06     y(n+1) = y(n) + h*f(t(n),y(n));
07     t(n+1) = t(n) + h;
08     n = n+1;
09 end
```

Let us take some time to understand this loop. We initialize our counter, `n`, at 1 (line 04). Remember that in line 02 we initialize the value of `t(1) = 0;`. We want to calculate the solution in increments of `h=0.00625` from 0 to 1. Thus we will calculate values for y until `t(n)>=1`. Now the first line in the loop, line 06, is the Euler method update. The next one, line 07, is the update to our time step. Finally, do not forget the last line, line 08, where we update our counter. Forgetting this line will lead to a never ending loop ...

We now see the power of such implementation. We can solve any ODE by simply changing the definition of `f` (and being careful about the initial condition), which is a line of code (line 19), and we can change the numerical solver we use by changing the update rule defined in the first line of code of the loop (line 06).

In Figure 20 we see a plot of the true solution against the numerical solution with 16 intermediate points.

We can clearly see how the numerical solution is worse as we move away from the origin. If we divide h by 10, and thus multiply the number of points used by 10, we can see in Figure 21 that the numerical solution is now very close to the real solution, and this with only 160 points

As we can see from these examples, the error produced by Euler method is too big, and thus better numerics where developed. A family of better numerics is that of the Runge-Kutta methods that we will study later.

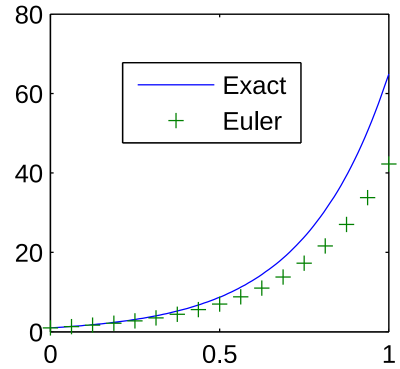


Figure 20: Euler method with $h = 0.0625$ from 0 to 1

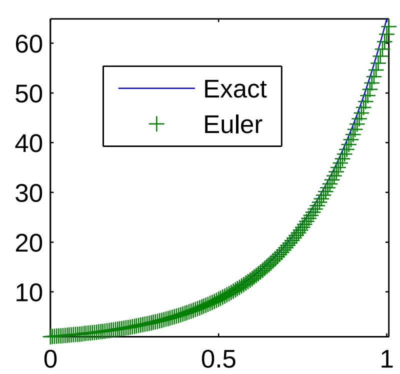


Figure 21: Euler method with $h = 0.00625$ from 0 to 1

9.3 Accuracy of the Euler method

Broadly speaking, accuracy is a measure of the error in a numerical scheme. The question we want to answer is the following. Assume we want the error in the numerical solution to be smaller than a prescribed value $\epsilon > 0$. How do we set the time step h in order to achieve such error?
First off, we need to clarify what we mean by *error* in the numerical method. As we mentioned previously, there are two types of errors we can consider

- single step/local error; and
- global error.

We begin with a treatment of the single step error for the Euler method.

1. Single Step/Local Error

   The single step error is defined as the error the numerical method makes *at the first step*. Thus we have that

   $$\epsilon_{\text{local}} = |y(t_1) - y_1| \tag{41}$$

   Recall that $y_1 = y_0 + hf(t_0, y_0)$ and thus we can write an explicit formula for the single step error in Euler's formula:

   $$\epsilon_{\text{local}} = |y(t_1) - y_0 - hf(t_0, y_0)| \tag{42}$$

   Hence we can see that, as we can advancing in time and integrating, the errors accumulate. We expect the final error to be bigger than the single step error.

   Now, how can we get an estimate for ϵ_{local}?

   - A first solution would be to select various values of h and plot $\log(\epsilon_{\text{local}})$ against $\log(h)$. We would observe a roughly straight line, thus giving us the relation between the local error and the time step.
   - Use the Taylor expansion of y.

   Recall that the Taylor expansion of y around t_0 is given by

   $$y(t_1) = y(t_0) + (t_1 - t_0)\frac{dy}{dt}(t_0) + \frac{(t_1 - t_0)^2}{2}\frac{d^2y}{dt^2}(t_0) + ... \tag{43}$$

   In our setting, we have that

   - The difference $t_1 - t_0$ is the time step h.
   - The value at t_0 is the initial condition: $y(t_0) = y_0$.
   - The derivative of y at t_0 is given by the ODE, thus $\frac{dy}{dt}(t_0) = f(t_0, y_0)$.

We conclude that

$$\begin{aligned}\pm\epsilon_{\text{local}} &= y(t_1) - y_0 - hf(t_0, y_0)\\ &\approx y(t_0) + (t_1 - t_0)\frac{dy}{dt}(t_0) + \frac{(t_1 - t_0)^2}{2}\frac{d^2y}{dt^2}(t_0) - y_0 - h\frac{dy}{dt}(t_0)\\ &\approx \frac{h^2}{2}\frac{d^2y}{dt^2}(t_0).\end{aligned}$$

We conclude that ϵ_{local} is proportional to h^2.

However, this only deals with what happens after one step of the method. We would like to know what happens after a large number of such steps. Thus we turn our attention to a more comprehensive notion of error.

2. Global Error

The global error, ϵ_{global}, is a measure of the error across the entire interval (from t_0 to t_n). As a first approximation, we assimilate the global error to the sum of all single step errors. Given that there are n such steps, we have that the global error is proportional to nh^2. This is more compactly written using the big "O" notation:

$$\epsilon_{\text{global}} = O(nh^2). \tag{44}$$

But the number of steps n is related to the time step h. Since $t_{i+1} = t_i + h$, we have that

$$n = \frac{t_n - t_0}{h} = \frac{t_{\text{final}} - t_{\text{initial}}}{h}$$

But t_{final} and t_{initial} are both constants, thus from Equation 44, we get

$$\epsilon_{\text{global}} = O\left(\frac{t_{\text{final}} - t_{\text{initial}}}{h}h^2\right) = O(h).$$

The global error is proportional to h. We also say that the Euler method is of **order 1**.

What does this tell us about the Euler method?

- It is not a very accurate method! Say we want $\epsilon_{\text{global}} \approx 10^{-6}$. Then the method would require roughly one million steps!
- If we have a given value of ϵ_{global} and h, we can calculate how to scale h in order to get a prescribed value for the global error.

Example 15 Assume that $h = 0.1$ and that global error is $\epsilon_{\text{global}} = 5\%$. If we want the error to be at most 1%, then we need to divide the step length by 5. Thus we need

$$h \leq \frac{0.1}{5} = 0.02$$

Example 16 Let's solve the ODE $y' = 1 - t + 4y$ with $y(0) = 1$. We solve this ODE using the Euler method using several values of h; we get the following results for the error:

$$
\begin{aligned}
h &= 10^{-2} & \epsilon_{\text{global}} &= 4.86 \\
h &= 10^{-3} & \epsilon_{\text{global}} &= 5.15\ 10^{-1} \\
h &= 10^{-4} & \epsilon_{\text{global}} &= 5.19\ 10^{-2} \\
h &= 10^{-5} & \epsilon_{\text{global}} &= 5.2\ 10^{-3}
\end{aligned}
$$

In the previous table, given that from line to line we divide the step length by 10, we expect the global error to be divided by 10 as well. As we can see, we need h to be small enough for this to happen. The accuracy results are *asymptotic* results (i.e., valid for small h).

9.4 Stability of the Euler Method

This is a difficult concept but a very important one. There are roughly two types of numerical methods:

1. Explicit methods. An example is the Euler method we presented at the beginning of this lecture. They are usually easy to program but might be unstable.

2. Implicit methods. They are harder to program but are usually *unconditionally stable*. Some problems require unconditional stability.

Let us explore the stability concept through the Euler method. We stick to a simple analysis that is relevant to the study of a single ODE (vs. a system of coupled ODEs).

Example 17 Euler Method is Unstable

Let $\alpha > 0$ be given. We consider the following equation

$$y'(t) = \alpha \left(y^2(t) - y^3(t) \right). \tag{45}$$

This equation is a good model for the growth of the radius of a flame. It is a separable equation, so let's try solving it using separation of variables. We can rewrite the equation as

$$\frac{dy}{y^2 - y^3} = \alpha \, dt$$

To integrate the left hand side, we use partial fractions to get

$$\frac{1}{y^2 - y^3} = \frac{1}{y} + \frac{1}{y^2} + \frac{1}{1-y}$$

which leads to

$$\log|y| - \log|1-y| - \frac{1}{y} = \alpha t + c$$

we take the exponential and get

$$\frac{y}{|1-y|}e^{-\frac{1}{y}} = c_0 e^{\alpha t}.$$

Getting an explicit solution for y does not seem possible; let us obtain a solution using the Euler method and the Matlab `ode45` solver.
We set $\alpha = 100$, $t_{\text{initial}} = 0$, $t_{\text{final}} = 2$ and $y(t_{\text{initial}}) = y(0) = 10^{-3}$. We plot the solution from `ode45` and that of the Euler method for various values of h.

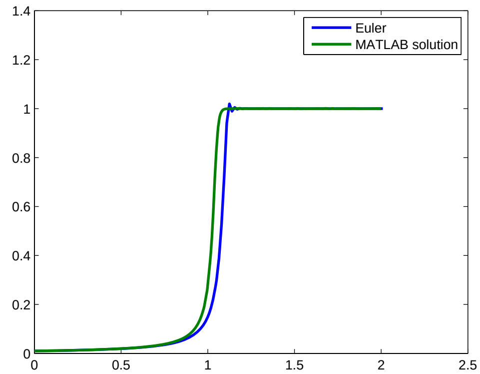


Figure 22: Comparing Euler's method solution to `ode45` solution for $h = 0.015$

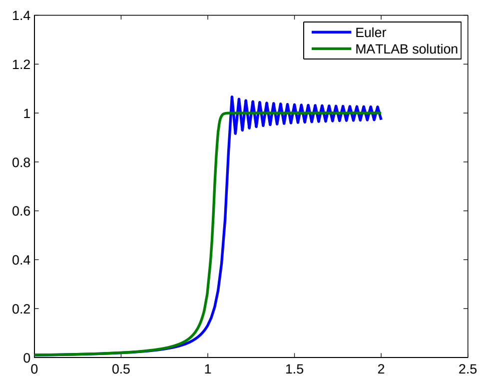


Figure 23: Comparing Euler's method solution to `ode45` solution for $h = 0.02$

As we can see, as h increases, we get a degradation of the accuracy. But when $h > 0.02$, we start getting very large oscillations in the solution and they do not seem to go away. The method has become unstable. In this case, the solution remains bounded, but for other equations, the solution may even grow unbounded!
The code used to produce Figures 22 to 25 is the following.

```
function stability_euler
```

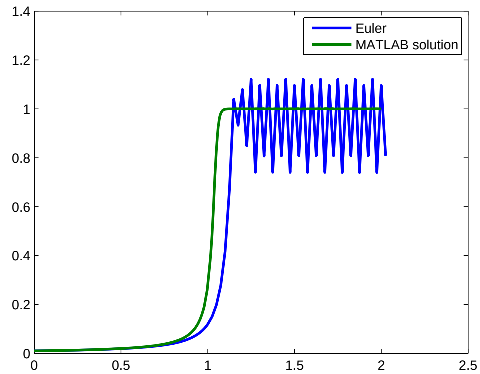


Figure 24: Comparing Euler's method solution to `ode45` solution for $h = 0.025$

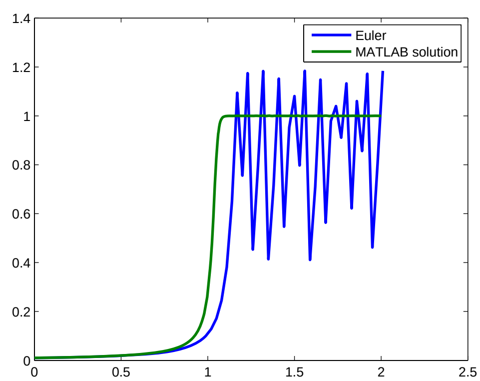


Figure 25: Comparing Euler's method solution to `ode45` solution for $h = 0.03$

```
y(1) = 1e-2; t(1) = 0;
h = 0.01;
n=1;
while t(n) < 2
    y(n+1) = y(n) + h*f(t(n),y(n));
    t(n+1) = t(n) + h;
    n = n+1;
end
[T,Y] = ode45(@f,[0 2],y(1));
plot(t,y,T,Y,'Linewidth',2)
legend('Euler','MATLAB solution')
end

function yp = f(t,y)
alpha = 1e2;
yp = alpha*(y^2-y^3);
end
```

Let us briefly comment on the command `ode45`. The first parameter is the function `f` with defines the ODE: $y' = f(t, y)$. Here it is passed as an input variable. Precisely, we say that we pass a function handle to `ode45`; this is the meaning of the "@". If our ODE was defined by a function `g`, we would use: `ode45(@g,...` The other arguments are the beginning and end time (0 and 2) and an initial value for the solution y.

Can we predict the values of h for which the Euler method is stable? Consider the following ODE

$$y'(t) = f(y) \tag{46}$$

where we have assumed that f does not depend on t. It is very difficult to study the stability of a numerical method in the general case, so we look at the stability near a *stable equilibrium point*, that we call y_e. Let us perform a Taylor expansion of f around y_e:

$$f(y) = f(y_\text{e}) + (y - y_\text{e})f'(y_\text{e}) + ... \tag{47}$$

But given that y_e is an equilibrium point, we have that $f(y_\text{e}) = 0$. Furthermore, y_e is a *stable* equilibrium point. Therefore, when $y > y_\text{e}$, the solution should tend to y_e again. Thus y' should be negative and we must have that $f'(y_\text{e}) < 0$. The same conclusion is reached if we consider $y < y_\text{e}$.
Hence we conclude that $f'(y_\text{e}) < 0$. We now approximate Equation 46 with

$$y' = (y - y_\text{e})\, f'(y_\text{e}) \tag{48}$$

Let's define the following change of variables $z = y - y_\text{e}$. By linearity of derivation, we have that $z' = y'$. Further, let $\lambda = f'(y_\text{e})$; we have that $\lambda < 0$. We can now define our *model* *problem*:

$$z' = \lambda z. \tag{49}$$

We know that the solutions to such equation are of the form $z(0)\exp(\lambda x)$. Given that $\lambda < 0$, we see that requiring y_{e} to be a *stable* equilibrium point ensures that the exact solution to the model problem stays bounded. This is always to be expected as our stability criterion for a numerical method is whether the numerical solution stays bounded or not.

Example 18 Flame Model

In the flame model ODE 45, we have that $f(y) = \alpha(y^2 - y^3)$. It can be shown that $y_{\mathrm{e}} = 1$ is a stable equilibrium point. Now we have that $f'(y) = \alpha(2y - 3y^2)$, thus we have that $f'(y_{\mathrm{e}}) = -\alpha$. Recall that $\alpha > 0$, thus the equilibrium point $y_{\mathrm{e}} = 1$ is indeed stable.

Thus the model problem here is

$$z' = \lambda z$$

where $z = y - 1$, and $\lambda = -\alpha$.

It is important to understand that the model problem is

- simple and thus tractable; and
- it is a good approximation for a stability analysis around a stable equilibrium point.

Stability. The stability criterion we will use is as follows: we require the numerical solution to stay bounded for the model problem 49.

Recall that the general iterate of the Euler method is defined as

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{50}$$

where $h = t_{n+1} - t_n$ and $y_0 = y(t_0)$. We write this equation for our model problem, and get

$$z_{n+1} = z_n + h\lambda z_n. \tag{51}$$

Let us see what z_1 is:

$$z_1 = (1 + \lambda h)z_0.$$

We do it again and get

$$z_2 = (1 + \lambda h)z_1 = (1 + \lambda h)(1 + \lambda h)z_0 = (1 + \lambda h)^2 z_0.$$

By induction, we can prove that, for all integers $k \geq 0$,

$$z_k = (1 + \lambda h)^k z_0. \tag{52}$$

Given that we want z_k to be bounded for all k, a necessary and sufficient condition is

$$|1 + \lambda h| \leq 1$$

Since $\lambda < 0$ and $h > 0$ this condition is equivalent to $-1 < 1 + \lambda h \Rightarrow (-\lambda)h < 2$. Thus the general condition for the Euler method to be stable for the model problem is

$$h \leq \frac{2}{|\lambda|}. \tag{53}$$

In our example we had that $\lambda = -\alpha = -100$, thus the Euler method is stable near $y = 1$ for $h \leq 0.02$. Note that our analysis is approximate and valid only near $y \sim 1$. This is why when $h > 0.02$ we do not necessarily see a diverging solution. Nevertheless we observed unphysical large oscillations when $h > 0.02$, confirming our analysis.
The stability analysis can be summarized as follows.

- Only valid near a stable equilibrium point y_e.
- $\lambda = f'(y_e)$.
- The upper bound on h is $2/|\lambda|$.

Example 19
Assume that the ODE we consider is $y' = -5y$. We have that $y_e = 0$ is a stable equilibrium point. We also have that $f'(y_e) = -5$. We thus obtain an upper bound for the time step:

$$h \leq \frac{2}{5}.$$

We now discuss the relevance of stability analysis for a numerical method. What we learn from the stability analysis of the Euler method is that

- Warning!! Numerical solutions can blowup.
- We can quantify how large the time step h can be in order for the method to be stable.

Even if the analysis is simple, it works well in practice. In molecular dynamics we model systems using non-linear ODEs. Typically, systems are far from equilibrium points. Nevertheless, if we use a time step consistent with what the linear stability analysis yields, things usually work fine.
It is also important to know that there are several fields where accuracy is less important than stability. For instance, in biology or chemistry the models used represent already approximations. Further, models tend to be quite complicated, thus solving the underlying equations is expensive. All this leads scientists to look for time steps as large as possible (thus reducing the cost of solving the equations). This is exactly what the stability analysis yields: an upper bound on how large the time step can be in order to have a stable method. For the mathematical development of numerical methods, stability analysis also provides a very useful metric to compare numerical methods.

10 Backward Euler Method

We will not focus on the code necessary to implement the backward Euler method as it can be quite complicated. Instead we simply state that the backward Euler method is given by

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \tag{54}$$

The key point is that given y_n and t_{n+1}, we need to solve for y_{n+1} since it appears on both sides of this equation. This is usually done using numerical methods such as Newton-Raphson.
From the general iterate for the backward Euler method, we can conclude that it has the same accuracy as the Euler method. We now focus on its stability.
We use the same model problem as in the Euler method, namely that $z' = \lambda z$, $\lambda < 0$, $z(0) = z_0$. We first calculate z_1:

$$\begin{aligned} &z_1 = z_0 + h\lambda z_1 \quad \text{we collect all } z_1 \text{ terms to the right hand side and get} \\ &z_1 = \frac{1}{1-\lambda h} z_0. \end{aligned}$$

We can again prove by induction that, for all integers $n \geq 0$,

$$z_n = \left(\frac{1}{1-\lambda h}\right)^n z_0. \tag{55}$$

Given that we want z_n to be bounded for all n, a necessary and sufficient condition is

$$\frac{1}{|1-\lambda h|} \leq 1 \tag{56}$$

but we know that $\lambda h < 0$, thus we have that $1 < 1 - \lambda h$, which implies that the inequality 56 is always satisfied! This is equivalent to saying that the backward Euler equation is **unconditionally stable**.

Summary of the differences between implicit and explicit methods.

- An explicit method is easy to implement but can become unstable.
- In an implicit method, we need to solve numerically for the iterate at each time step. It can be computationally expensive, but it is unconditionally stable.

Being able to understand the differences between implicit and explicit methods is very important. In PDEs, not only do we need to discretize time, we also need to discretize space. If we are working in a two dimensional space, we discretize space in triangles. Usually, the space discretization, or *mesh*, is not uniform. Further, the time step is related to the shape of these triangles. If we have even a few very distorded triangles in our mesh, the maximum time step implementable so that an explicit method is stable would be very small. This is regardless of all other triangles being well proportioned.

Thus, when solving a PDE with an explicit method, an important part of the effort is to optimize the mesh in order to have as large a time step as possible. If, on the other hand an implicit method is used, stability is no longer an issue. Much larger time steps can be used. A lot of effort is then placed on efficiently implementing the implicit method so that it is not computationally too expensive to calculate each iterate.

11 Runge-Kutta Methods

11.1 Numerical algorithms

The global error of the Euler method is $O(h)$, where h is the time-step taken. This is not good enough for most engineering problems. Can we design numerical methods that are more accurate than the Euler method without being computationally too expensive?
Recall that the problem we are interested in solving is

$$y' = f(t, y), \quad \text{for } t \in (t_{\text{initial}}; t_{\text{final}}) \tag{57}$$

$$y(t_{\text{initial}}) = y_0. \tag{58}$$

The Runge-Kutta method is a general class of numerical integration methods that works in the following way. Given the approximate value of the solution to the Equation 57 at time $t_n \in (t_{\text{initial}}; t_{\text{final}})$, we calculate the approximate value of the solution at time $t_{n+1} = t_n + h$ using the following equation

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + ... + w_m k_m) \tag{59}$$

and the parameters $\{w_1, ..., w_m, k_1, ..., k_m\}$ are such that

1. for all i, $w_i \geq 0$;
2. the sum of all w_i is one, i.e. $\sum_{i=1}^{m} w_i = 1$;
3. for each i, k_i is the value of $f(t, Y)$ evaluated at a suitable t and Y; and
4. the resulting numerical method has a single step error of $O(h^{m+1})$.

We can rephrase the fourth condition by saying that we choose the parameters such that Equation 59 matches the Taylor expansion of the solution at t_{n+1}, up to the $(m+1)^{th}$ term. There exist several Runge-Kutta methods for a given order m, thus we are just going to present a couple of examples.

Example 20 Second Order Runge-Kutta or Improved Euler Method.
In this method, the update on the value of y is defined as

$$y_{n+1} = y_n + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right), \quad \text{where} \tag{60}$$

$$k_1 = f(t_n, y_n), \quad \text{and} \tag{61}$$

$$k_2 = f(t_n + h, y_n + hk_1). \tag{62}$$

We can see that, in the definition of k_2, we use $y_n + hk_1$, which is the Euler method's update. By performing a Taylor expansion, we can show that this method is indeed a *second order* method, i.e. the global error is $O(h^2)$. We also can prove that the local error/single step error is $O(h^3)$ for the Taylor expansion is matched up to the second order.

Example 21 Runge-Kutta 4.
RK4 is the standard integration method used. We present the standard formulation, though Matlab uses a different one (i.e. the parameters used are different). Proving that RK4 is indeed a fourth order method requires complex calculations, so we will just assume it is true. In this method, the update on the value of y is defined as

$$y_{n+1} = y_n + h\left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right), \quad \text{where} \tag{63}$$

$$k_1 = f(t_n, y_n), \quad \text{and} \tag{64}$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \quad \text{and} \tag{65}$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \quad \text{and} \tag{66}$$

$$k_4 = f\left(t_n + h, y_n + hk_3\right). \tag{67}$$

The single step error of this method is $O(h^5)$, while its global error is $O(h^4)$, thus it is a fourth order method.
Here is the matlab code:

```
function [t,y] = RK4(f,tspan,y0,h)
% Runge-Kutta 4 method
% Input arguments:
% f: function taking as input t and y and returning dy/dt
% tspan: vector containing the initial time tspan(1) and final time
%        tspan(2)
% y0: initial condition y(0)
% h: time step

y(1) = y0; t(1) = tspan(1); n=1;
while t(n) < tspan(2)
    k1 = f(t(n),y(n));
    k2 = f(t(n)+1/2*h,y(n)+1/2*h*k1);
    k3 = f(t(n)+1/2*h,y(n)+1/2*h*k2);
    k4 = f(t(n)+h,y(n)+h*k3);
    y(n+1) = y(n) + h*(1/6*k1+1/3*k2+1/3*k3+1/6*k4);
    t(n+1) = t(n) + h;
    n = n+1;
```

```
end
end
```

The code for RK2 is very similar.

Computational efficiency of numerical integrators. When evaluating the cost of a numerical method, usually the most expensive part is evaluating the function f. In practical problems, f can be very difficult to evaluate, and thus, for a given accuracy, we compare numerical integrators by considering the number of function evaluations necessary to achieve the specified accuracy.
We can see that, for each time step, RK2 evaluates f twice, whereas RK4 evaluates f four times. Thus RK4 is twice as expensive as RK2 per step. But the global error of RK2 is only the square root of the global error of RK4 (recall that global errors are numbers less than 1); thus many more steps are needed in RK2 and consequently RK2 usually ends up being more expensive than RK4.

11.2 Error analysis

We use the following code to compare the performance of Euler, RK2 and RK4.
File "`comparison.m`"

```
function comparison
% Comparison of Euler, RK2 and RK4 for the following problem:
% dy/dt = 2*t*y
% y(t=1) = 1
% Exact solution: e^(t^2-1)

    h = 0.1;
    tspan(1) = 1;
    tspan(2) = 1.5;
    y0 = 1;
    [t,y_eul] = Euler(@f,tspan,y0,h);
    [t,y_rk2] = RK2(@f,tspan,y0,h);
    [t,y_rk4] = RK4(@f,tspan,y0,h);
    fprintf('\nt_n       Euler       RK2         RK4         Exact\n');
    for k=1:length(t)
        fprintf('%4.2f      %6.4f      %6.4f      %6.4f..
              %6.4f\n',t(k),y_eul(k),y_rk2(k),y_rk4(k),exact(t(k)));
    end
end

function yp = f(t,y)
```

```
    yp = 2*t*y;
end

function y = exact(t)
    y = exp(t.^2-1);
end
```

We look at the values of the function for different values of t_n. We do so for Euler, RK2 and RK4 and compare it to the exact solution

t_n	Euler	RK2	RK4	Exact
1.00	1.0000	1.0000	1.0000	1.0000
1.10	1.2000	1.2320	1.2337	1.2337
1.20	1.4640	1.5479	1.5527	1.5527
1.30	1.8154	1.9832	1.9937	1.9937
1.40	2.2874	2.5908	2.6116	2.6117
1.50	2.9278	3.4509	3.4902	3.4903

Table 1: Estimates of the solution for different methods

As we can see from Table 1, RK4 is identical to the exact solution up to 4 decimal places. We can also see that RK2 is not too bad, and that Euler can only give a guarantee on the first digit at time 1.50.
We now turn our attention to the behavior of the global error for RK4 as a function of h. We use the same ODE and initial conditions are the case above. We calculate the error for RK4 for $h = 0.10$ and $h = 0.05$, and we get the following results for the global error: As we

h	Global error
0.10	6.12e-005
0.05	3.78e-006

Table 2: Global error for RK4

can see from Table2, if we divide the time step by 2, the error is roughly divided by 16.
Next, we compare RK4 and `ode45`:

```
function rk4_ode45
% ODE is:
% dy/dt = -y+10*sin(3*t)
% y(0) = 0
% Exact solution: -3 cos(3t) + sin(3t) + 3 exp(-t)

    t = linspace(0,2,1e5);
```

```
    h = 0.01;
    tspan(1) = 0; tspan(2) = 2; y0 = 0;
    [t_rk4,y_rk4] = RK4(@f,tspan,y0,h);
    [t_ode45,y_ode45] = ode45(@f,tspan,y0);

    subplot(1,2,1)
    plot(t,exact(t),t_rk4,y_rk4,t_ode45,y_ode45)
    legend('Exact solution','RK4','ode45')
    subplot(1,2,2)
    semilogy(t_rk4,abs(y_rk4-exact(t_rk4)),t_ode45,...
             abs(y_ode45-exact(t_ode45)))
    legend('RK4','ode45')
end

function yp = f(t,y)
    yp = -y+10*sin(3*t);
end

function y = exact(t)
    y = -3*cos(3*t)+sin(3*t)+3*exp(-t);
end
```

The result is a plot comparing the solutions provided by both methods and their respective local errors. We can see those results in Figure 26. Note that the behavior of the error in RK4 is very different from that of `ode45`. In `ode45`, at each step, the time step is chosen so that the single-step error is $\sim 10^{-5}$. This is called *adaptive time-stepping*. Typically, smaller time steps are chosen when the solution changes very quickly, and larger time steps if it is not changing too much. On the contrary in RK4, we chose a fixed time step.

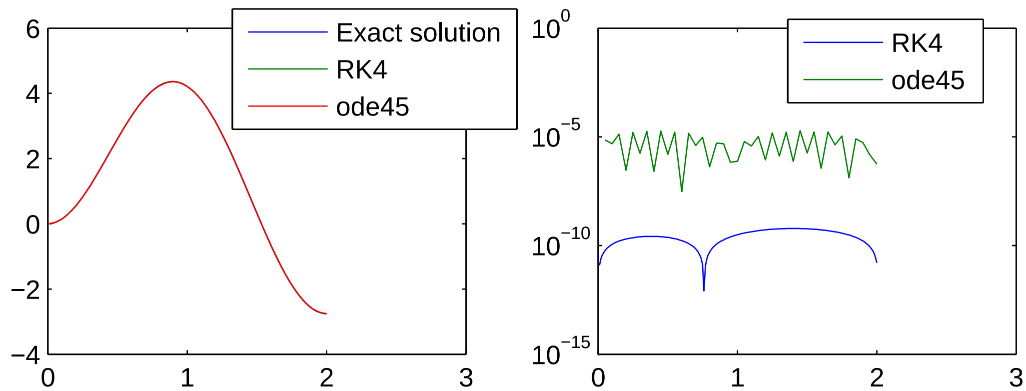


Figure 26: Left: comparison of solutions from RK4 and `ode45`; Right: their local errors.

12 Multi-Step Methods

We focus on first order ODEs as before.

$$y'(t) = f(t, y), \quad \text{subject to} \tag{68}$$

$$y(t_0) = y_0. \tag{69}$$

There are, broadly speaking, two classes of techniques to numerically solve ODEs.

- Multi-stage methods. These include all the RK family of methods. Multi-stage methods start from the initial conditions and produce, iteratively, new positions. More abstractly, if we are given y_n, the value of the approximation to the solution at t_n, we can calculate the value y_{n+1}, the approximation to the solution at t_{n+1}.

  We call them multi-stage because, at any given iteration, we evaluate several times (or stages) the function f. In RK4, there are 4 stages. The problem with multi-stage methods is that they can become expensive if we need a highly accurate solution.

- Multi-step methods. We are given all the (t_i, y_i) up to n. The idea behind multi-step methods is to use more data points than just (t_n, y_n) to calculate (t_{n+1}, y_{n+1}).

Multi-step methods thus look more like an extrapolation problem, where we want to infer the value of y_{n+1} that "best fits" given all the history of values (t_i, y_i) up to n. The most common family of integrators for multi-step methods the *Adams* family of integrators.

12.1 Adams-Bashforth

To design multi-stage methods, as in our study of RK methods, we design *stages* so that we approach the Taylor expansion of the solution up to a given order. In multi-step methods we use a different approach.
We start by noting that the exact solution satisfies the following equation.

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} y'(t)dt. \tag{70}$$

We are going to approximate the integral

$$I(t_n, t_{n+1}) = \int_{t_n}^{t_{n+1}} y'(t)dt. \tag{71}$$

in two steps.

- *Step 1*: Find a polynomial $P(t)$ that approximates $y'(t) = f(t, y)$ in the vicinity of t_n.

- *Step 2*: Approximate $I(t_n, t_{n+1})$ with

$$I_n = \int_{t_n}^{t_{n+1}} P(t)dt$$

and calculate the next iterate

$$y_{n+1} = y_n + I_n.$$

In order to see how we calculate $P(t)$, we are going to treat explicitly the case when we approximate $f(t, y)$ with a polynomial of degree at most 1. Thus we need to determine A and B such that $P(t) = At + B$ interpolates y' given its value at t_{n-1} and t_n. Hence we get that

$$P(t_{n-1}) = f(t_{n-1}, y_{n-1})$$
$$P(t_n) = f(t_n, y_n)$$

We can see in Figure 27 pictorially what we are doing.

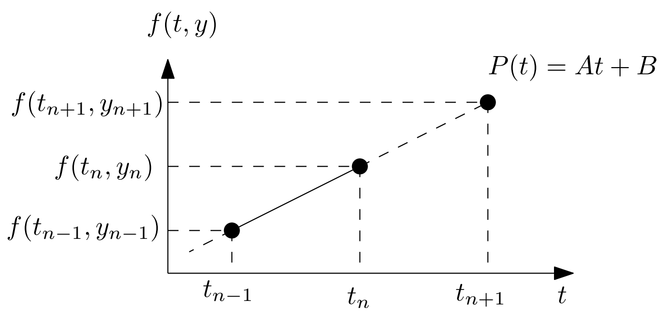


Figure 27: Linear interpolation

It is now clear how one would proceed to get higher order methods: we could use the values from t_{n-2}, t_{n-1} and t_n to get a second order interpolation polynomial, and thus increase the accuracy of the method.
It is important to remember that the polynomial we calculate to produce I_n is going to be different from the polynomial used to calculate, say, I_{n+1}.
We now calculate explicitly the values for A and B. In order to simplify notation, we set

$$f(t_n, y_n) = f_n.$$

Thus we have that

$$At_{n-1} + B = f_{n-1}$$
$$At_n + B = f_n.$$

If we assume that the time step h is constant, we have that

$$A = \frac{f_n - f_{n-1}}{h}$$
$$B = \frac{f_{n-1}t_n - f_n t_{n-1}}{h}$$

We now calculate

$$I_n = \int_{t_n}^{t_{n+1}} P(t)dt.$$

$$\begin{aligned}
I_n &= \int_{t_n}^{t_{n+1}} (At+B)dt \\
&= \frac{A}{2}\left(t_{n+1}^2 - t_n^2\right) + B\left(t_{n+1} - t_n\right) \\
&= \frac{f_n - f_{n-1}}{2h}(t_{n+1}+t_n)(t_{n+1}-t_n) + \frac{f_{n-1}t_n - f_n t_{n-1}}{h}h \\
&= \frac{f_n - f_{n-1}}{2}(t_{n+1}+t_n) + f_{n-1}t_n - f_n t_{n-1} \\
&= f_n\left(\frac{1}{2}(t_{n+1}+t_n) - t_{n-1}\right) + f_{n-1}\left(-\frac{1}{2}(t_{n+1}+t_n) + t_n\right) \\
&= \frac{1}{2}f_n(t_{n+1}+t_n-2t_{n-1}) - \frac{1}{2}f_{n-1}h \\
&= \frac{1}{2}f_n(t_{n+1}-t_n+t_n+t_n-2t_{n-1}) - \frac{1}{2}f_{n-1}h \\
&= \frac{1}{2}f_n(t_{n+1}-t_n+2t_n-2t_{n-1}) - \frac{1}{2}f_{n-1}h \\
&= \frac{3h}{2}f_n - \frac{h}{2}f_{n-1}.
\end{aligned}$$

We finally get the recurrence relationship

$$y_{n+1} = y_n + \frac{3h}{2}f_n - \frac{h}{2}f_{n-1}.$$

The local error for this method is $O(h^3)$, thus this is a second order method (the global error is $O(h^2)$).
It is not important to remember the algebra that got us the result, but rather to remember the concept behind the method.
We provide the formula for the fourth order method:

$$y_{n+1} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right].$$

The global error of this method is $O(h^4)$. (This can be obtained using a cubic polynomial to approximate y'). Fourth order is usually a good order to shoot for as it is accurate enough and not too expensive.
Key observations:

1. In RK2, we need 2 function evaluations per time step. In RK4, we need 4. The higher the order of the method, the higher the number of function evaluations needed per time step. In Adams Bashforth method, to evaluate y_{n+1}, we need the values of f_n, f_{n-1}, f_{n-2} and f_{n-3}. But, in order to evaluate y_n, we used f_{n-1}, f_{n-2}, f_{n-3} and f_{n-4}. Thus, in order to evaluate y_{n+1}, we only need one extra function evaluation!

2. But, how do we initialize Adams Bashforth method? This is not a self-starting method! We need another technique, of the same order or higher (here, say, RK4), to initialize the Adams Bashforth method.

Thus, if we use RK4 to initialize the Adams Bashforth method, we compute only the first 3 iterates of y_n so that we can calculate f_1, f_2 and f_3. Then, using the initial conditions, we calculate f_0, and we can now use the Adams Bashforth method exclusively.

12.2 Adams Moulton Method

The Adams-Moulton method builds on the Adams Bashforth method and improves its accuracy. It is more costly but it is also substantially more accurate. Adams-Moulton method belongs to the *predictor-corrector* methods family.

- *Step 1*: Predictor step uses Adams Bashforth method.
- *Step 2*: Corrector step modifies the result from the predictor step to enhance accuracy.

Predictor Step

$$y^*_{n+1} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right].$$

Corrector Step

$$y_{n+1} = y_n + \frac{h}{24}\left[9f^*_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right]$$

where

$$f^*_{n+1} = f(t_{n+1}, y^*_{n+1}).$$

This is the method used in the function *ode113* from Matlab. *ode113* is an advanced solver which varies the time step and the order of the method.
In our case, we use a corrector step to decrease the value of the error but the order of the method is the same. There are instances of predictor-corrector integrators where the order increases after the corrector step.
We can see an implementation of the Adams-Moulton method in the file "`Multistep.m`". The results from a run of the script are summarized in Table 3

n	x_n	Starting y_n	Predicted	Corrected	Exact Value
0	0.0	1.000000			1.000000
1	0.2	1.021403			1.021403
2	0.4	1.091825			1.091825
3	0.6	1.222119			1.222119
4	0.8		1.425375	1.425543	1.425541
5	1.0		1.718083	1.718287	1.718282
6	1.2		2.119877	2.120127	2.120117
7	1.4		2.654911	2.655216	2.655200
8	1.6		3.352684	3.353057	3.353032
9	1.8		4.249228	4.249683	4.249647
10	2.0		5.388551	5.389107	5.389056
11	2.2		6.824405	6.825085	6.825013

Table 3: Results for the Adams-Moulton method

```
function multistep
% Script to demonstrate the implementation and use of the Adams-Moulton
% method.

    tspan = [0 2];
    h = 0.2;
    y0 = 1;

    [t,y_star,y] = adams_moulton(@yprime,tspan,y0,h);

    fprintf('\n n   x_n   Starting y_n  Predicted  Corrected  ..
    Exact Value\n');
    for k=1:4
        fprintf('%2i   %3.1f   %8.6f  %8.6f\n',k-1,t(k),y(k),exact(t(k)));
    end

    for k=5:length(t)
        fprintf('%2i   %3.1f  %8.6f   %8.6f   %8.6f\n',k-1,t(k),..
        y_star(k),y(k),exact(t(k)));
    end
end

function [t,y_star,y] = adams_moulton(f,tspan,y0,h)
```

```
% First compute the solution at a few points to get started
    [t,y] = ode45(f,0:h:3*h,y0);
    y_star = y; % Will contain the value at the end of the predictor step

    fn = f(t,y); % Stores the computed values of f so far

% Step forward using Adams-Moulton
    n = 4; % We already computed the first 3 steps
    while t(n) < tspan(2)
        t(n+1) = t(n)+h;
        % Predictor step
        y_star(n+1) = y(n) + h/24*(55*fn(n)-59*fn(n-1)..
         +37*fn(n-2)-9*fn(n-3));

        f_star = f(t(n+1),y_star(n+1));

        % Corrector step
        y(n+1) = y(n) + h/24*(9*f_star+19*fn(n)-5*fn(n-1)+fn(n-2));

        fn(n+1) = f(t(n+1),y(n+1));

        n=n+1;
    end
end

% The ODE we are solving for
function yp = yprime(t,y)
    yp = t+y-1;
end

% The exact solution
function y = exact(t)
    y = -t+exp(t);
end
```

In the previous implementation, we saw an new way to call the function *ode45*:

```
    [t,y] = ode45(f,0:h:3*h,y0);
```

It is useful to know that we can specify to *ode45*, via a vector (here `0:h:3*h`), the exact points where we want to approximate the value of the solution. Thus, to initialize the Adams-Moulton method, we need to calculate the value of the solution at h, $2h$ and $3h$.

13 Second Order ODEs

The study of second order ODEs is very important. Many mechanical and dynamical systems can be modeled using second oder ODEs. Unfortunately, second order ODEs are far more complicated to solve than first order ODEs. In fact, solving linear second order ODEs is already very complicated, so we won't cover non-linear ODEs.
While this can be viewed as a setback, it is possible to linearize any non-linear ODE when the solution makes small oscillations around an equilibrium point. Recall that this is how we came-up with the linear version of the second order ODE describing the pendulum's angle.
The general second order linear ODE we will consider can be written as follows.

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \tag{72}$$

Recall that, depending on $r(x)$, we call this equation

- *homogeneous* if $r(x) = 0$,
- *inhomogeneous* if $r(x) \neq 0$.

We now focus on homogeneous equations:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \tag{73}$$

To understand the profound differences with respect to the case of first order equations, let us focus on the pendulum's example again.
Example 22 Pendulum
Recall that if the angle, θ, the pendulum makes with respect to the vertical axis is small, the ODE resulting from Newton's second law is given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

We study a simpler version of this equation, namely

$$\frac{d^2y}{dx^2} + y = 0 \tag{74}$$

where we changed the variable and the function's name to stick to the notation from Equation 73.
Recall the derivatives of the sine and cosine functions:

$$\begin{aligned} \frac{d}{dx}[\cos(x)] &= -\sin(x) \quad ; \quad \frac{d}{dx}[\sin(x)] = \cos(x) \\ \frac{d^2}{dx^2}[\cos(x)] &= -\cos(x) \quad ; \quad \frac{d^2}{dx^2}[\sin(x)] = -\sin(x) \end{aligned}$$

thus we conclude that both $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$ are solutions to Equation 74. Given that the equation is linear we have that for any constants c_1 and c_1, $y(x) = c_1 \cos(x) + c_2 \sin(x)$ is a solution. In fact, it is possible to prove that $y(x) = c_1 \cos(x) + c_2 \sin(x)$ is the *general solution* to Equation 74.
We now see why second order equations are so different from first order ones. Here the general solution depends on *two* arbitrary constants instead of just one. Thus *two* initial conditions are needed in order to determine the particular solution. Here we use

$$y(0) = 1 \tag{75}$$

$$\frac{dy}{dt}(0) = 0 \tag{76}$$

The first equation yields $c_1 \cos(0) + c_2 \sin(0) = c_1 = 1$, while the second equation yields $-c_1 \sin(0) + c_2 \cos(0) = c_2 = 0$. Thus the *particular solution* to Equation 74 given initial conditions 75 and 76 is

$$y(x) = \cos(x).$$

We now state the main result of this subsection. In order to do so, we need to introduce the concept of *linear independence* of two functions.

Definition 1 Linear Dependence and Independence
We say that two functions $y_1(x)$ and $y_2(x)$ are *linearly dependent* if they are proportional to one another. Two functions that are not linearly dependent are said to be *linearly independent*.

Theorem 1. *Let y_1 and y_2 be solutions to Equation 73. If y_1 and y_2 are linearly independent, then*

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is the general solution where c_1 and c_2 are arbitrary constants.

We are not going to prove this theorem. However, we can easily see why a linear combination of two solutions gives a new solution. Assume y_1 and y_2 are solutions and consider $y = c_1 y_1 + c_2 y_2$. Let us show that y is a solution as well:

$$\begin{aligned} y'' + p\,y' + q\,y &= (c_1 y_1 + c_2 y_2)'' + p\,(c_1 y_1 + c_2 y_2)' + q\,(c_1 y_1 + c_2 y_2) \\ &= c_1 (y_1'' + p\,y_1' + q\,y_1) + c_2 (y_2'' + p\,y_2' + q\,y_2) = 0 \end{aligned}$$

The last step is equal to 0 because y_1 and y_2 are solutions of the equation. Can you see that this result is true for linear homogeneous equations only?

The solution process to find the general solution to linear, second order homogeneous ODEs is quite different from that of first order linear homogeneous ODEs. In the first order case, we essentially perform one integration. Here we need to get two independent solutions and

use Theorem 1. We will later see how to get linearly independent solutions for some second order linear ODEs.
Once we have two linearly independent solutions, in an *initial value problem*, we determine the constants c_1 and c_2 using initial conditions of the form given

$$y(0) = y_0 \tag{77}$$
$$\frac{dy}{dx}(0) = y'_0 \tag{78}$$

In this class we focus on initial value problems. Another very important class of problems is that of *boundary value problems*. These tend to be more complicated as they can have 0, 1 or multiple solutions. We focus on initial value problems as they tend to have a unique solution.
Example 23 A simple initial value problem
Assume that we need to solve the equation $\frac{d^2y}{dx^2} - y = 0$. We can immediately see that both $y_1(x) = \exp(x)$ and $y_2(x) = \exp(-x)$ are solutions to this equation. Moreover, y_1 and y_2 are linearly independent, thus the general solution to this ODE is given by

$$y(x) = c_1 e^x + c_2 e^{-x}.$$

We now consider the following initial conditions $y(0) = 1$ and $y'(0) = 2$. Thus we have the following two equations

$$c_1 e^0 + c_2 e^{-0} = c_1 + c_2 = 1$$
$$c_1 \frac{d}{dt}\left[e^x\right](0) + \frac{d}{dt}\left[e^{-x}\right](0) c_2 = c_1 - c_2 = 2$$

that we need to satisfy simultaneously. The solution to such system is

$$c_1 = \frac{3}{2}$$
$$c_2 = -\frac{1}{2}$$

which leads to the particular solution

$$y(x) = \frac{3}{2}e^x - \frac{1}{2}e^{-x}.$$

Figure 28 is a plot of the solution to this particular initial value problem.

14 Method of Reduction of Order

Recall that we are considering linear homogeneous second order ODEs:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

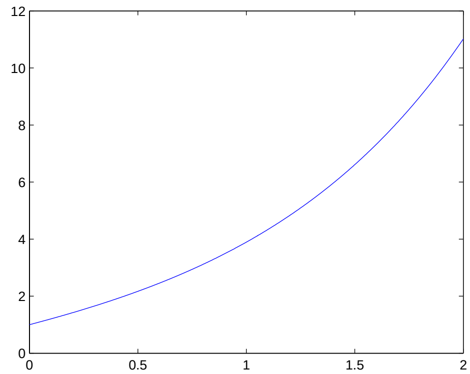


Figure 28: Particular solution to $y'' - y = 0$

Assume that you know a solution y_1 to Equation 73. To get the general solution we only need to find a second solution y_2 that is linearly independent from y_1.
This situation arises when solving second order linear homogeneous ODEs with constant coefficients. Later, we will develop a powerful method to get the general solution to such equations. Unfortunately there are cases where such technique only yields one solution. Using the method of reduction of order we get the second solution and thus the general solution.
The method of order reduction is such that, given a solution y_1, it produces a solution y_2 by solving a first order ODE derived from Equation 73 in a way similar to that of the integrating factor methods.
Given that $y_1 \neq 0$, any function $y_2(x)$ can be written as $y_2(x) = u(x)y_1(x)$. If we further impose that y_2 is a solution to Equation 73, we get that (for clarity, we exchange right hand side and left hand side)

$$0 = \frac{d^2y_2}{dx^2} + p(x)\frac{dy_2}{dx} + q(x)y_2 \tag{79}$$

Let us first calculate the first and second derivatives of y_2. The first derivative of y_2 is given by the product rule

$$\frac{dy_2}{dx} = \frac{d}{dx}\left[uy_1\right] = \frac{du}{dx}y_1 + u\frac{dy_1}{dx}.$$

We derive once more this expression to get the second derivative of y_2

$$\begin{aligned}\frac{d^2y_2}{dx^2} &= \frac{d^2}{dx^2}\left[uy_1\right] = \frac{d}{dx}\left[\frac{du}{dx}y_1 + u\frac{dy_1}{dx}\right]\\ &= \frac{d}{dx}\left[\frac{du}{dx}y_1\right] + \frac{d}{dx}\left[u\frac{dy_1}{dx}\right]\\ &= \frac{d^2u}{dx^2}y_1 + \frac{du}{dx}\frac{dy_1}{dx} + \frac{du}{dx}\frac{dy_1}{dx} + u\frac{d^2y_1}{dx^2}\\ &= \frac{d^2u}{dx^2}y_1 + 2\frac{du}{dx}\frac{dy_1}{dx} + u\frac{d^2y_1}{dx^2}\end{aligned}$$

We now replace the y_2 derivatives in Equation 79:

$$
\begin{aligned}
0 &= \frac{d^2y_2}{dx^2} + p(x)\frac{dy_2}{dx} + q(x)y_2 \\
0 &= \frac{d^2u}{dx^2}y_1 + 2\frac{du}{dx}\frac{dy_1}{dx} + u\frac{d^2y_1}{dx^2} + p(x)\left(u\frac{dy_1}{dx} + \frac{du}{dx}y_1\right) + q(x)uy_1 \\
0 &= \frac{d^2u}{dx^2}y_1 + 2\frac{du}{dx}\frac{dy_1}{dx} + u\frac{d^2y_1}{dx^2} + p(x)u\frac{dy_1}{dx} + p(x)\frac{du}{dx}y_1 + q(x)uy_1
\end{aligned}
$$

Let us gather together all terms multiplied by u, and u' and get

$$
0 = \frac{d^2u}{dx^2}y_1 + \frac{du}{dx}\left(2\frac{dy_1}{dx} + p(x)y_1\right) + u\left(\frac{d^2y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1\right)
$$

but y_1 is a solution of the ODE, thus the last term is zero. We finally get that $u(x)$ is such that

$$
\frac{d^2u}{dx^2}\,y_1 + \frac{du}{dx}\left(2\frac{dy_1}{dx} + p(x)y_1\right) = 0. \tag{80}
$$

Notice that this equation has no u in it. This suggests the following change of variable:

$$
w = \frac{du}{dx} \tag{81}
$$

Divide by y_1 to get the following equation

$$
\frac{dw}{dx} + w\left(2\frac{y_1'}{y_1} + p(x)\right) = 0 \tag{82}
$$

which is a first order separable ODE! We use the separation of variables technique and get

$$
\begin{aligned}
\frac{dw}{w} &= -\left(2\frac{y_1'}{y_1} + p(x)\right)dx \\
\log|w| &= -\int\left(2\frac{y_1'}{y_1} + p(x)\right)dx \\
\log|w| &= -\int\left(2\frac{y_1'}{y_1}\right)dx - \int p(x)dx \\
\log|w| &= -2\log|y_1| - \int p(x)dx + c
\end{aligned}
$$

we take the exponential and get

$$
|w| = \frac{1}{y_1^2}e^{c-\int p(x)dx}.
$$

As usual we change e^c to $\pm c_0$:

$$
w(x) = c_0\frac{e^{-\int p(x)dx}}{y_1^2}.
$$

But if y_2 is a solution to the ODE, so is cy_2 for any c; so we can always choose $c_0 = 1$. From Equation 81, we get the following expression for u

$$u(x) = \int \left(\frac{e^{-\int p(x)dx}}{y_1^2} \right) dx + c'$$

But recall that $y_2(x) = u(x)y_1(x)$. Again we can drop the integration constant c' since we are only looking for a solution linearly independent from y_1:

$$y_2(x) = y_1(x) \int \left(\frac{e^{-\int p(x)dx}}{y_1^2} \right) dx.$$

Note that the resulting function y_2 is linearly independent from y_1. Hence we can now give the general solution to Equation 73:

$$\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x) = y_1(x) \left[c_1 + c_2 \int \left(\frac{e^{-\int p(x)dx}}{y_1^2} \right) dx \right]} \tag{83}$$

This is a very important formula which you should put in your sheet of formulas for the exam.
Recall that this formula can only be applied to an ODE in its **standard form**.
The reduction of order method is very useful when:

- some known technique only yields one solution. We'll develop this point later.
- when the ODE has one "obvious" solution.

Example 24 Euler-Cauchy Equation
We want to solve the following second order linear ODE

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 0. \tag{84}$$

First note that $y_1(x) = x^2$ is a solution to this equation:

$$\begin{aligned} x^2 \frac{d^2y_1}{dx^2} - 3x \frac{dy_1}{dx} + 4y_1 &= x^2 \frac{d^2x^2}{dx^2} - 3x \frac{dx^2}{dx} + 4x^2 \\ &= x^2(2) - 3x(2x) + 4x^2 \\ &= 2x^2 - 6x^2 + 4x^2 \\ &= 0 \end{aligned}$$

We next put Equation 84 into standard form and get

$$\frac{d^2y}{dx^2} - \frac{3}{x} \frac{dy}{dx} + \frac{4}{x^2} y = 0. \tag{85}$$

thus $p(x) = -\frac{3}{x}$, which leads to

$$\begin{aligned} u(x) &= \int \left(\frac{e^{-\int \frac{-3}{x} dx}}{x^4} \right) dx \\ &= \int \left(\frac{e^{3\int \frac{1}{x} dx}}{x^4} \right) dx \\ &= \int \left(\frac{e^{3\log|x|}}{x^4} \right) dx \\ &= \int \left(\frac{|x|^3}{x^4} \right) dx \\ &= \int \left(\frac{1}{|x|} \right) dx \end{aligned}$$

We can remove the absolute value given that we can multiply u by any constant and it still yields a solution to our problem, thus we have that

$$u(x) = \log|x|$$

which yields

$$y_2(x) = x^2 \log|x|.$$

It is important to note that we cannot remove the absolute value in the expression for y_2 as that would imply that y_2 is only defined for positive values of x. It is also important to note that, at $x = 0$, complicated things happen. We thus would need initial conditions to be specified at a different point than $x = 0$, say $x = 1$.
One can now see that, even though one solution was easy to find, the second one can be very complicated. This is why the reduction of order method can be very useful.
The general solution for Equation 84 is finally given by

$$y(x) = x^2 \left(c_1 + c_2 \log|x|\right).$$

15 Second order ODEs with constant coefficients

In this section we focus on the study of second order linear homogeneous ODEs with constant coefficients. The general form for such equations is given by

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0. \tag{86}$$

It is important to review circular functions and complex numbers. We refer you to the handout 9.5 on the subject. The fundamental idea behind the study of linear ODEs with constant coefficients is better understood through the following example.

Example 25 Simple ODE with Constant Coefficients
Consider the following ODE.

$$\frac{dy}{dx} = \lambda y.$$

The solutions to this first order ODE are given by $y(x) = c\exp(\lambda x)$.
Are the solutions to second order ODEs related to exponential functions? Let us find out if there are solutions to Equation 86 of the form

$$y(x) = e^{rx}.$$

If such a solution exists, then we have that Equation 86 is equivalent to

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Divide by $\exp(rx)$ to obtain the **characteristic equation** of Equation 86:

$$ar^2 + br + c = 0. \tag{87}$$

This is excellent! Let r_1 and r_2 be the solutions to the characteristic equation. We have three different cases:

1. $b^2 > 4ac$. In this case, $r_1 \neq r_2$ and both roots are real. Thus we have that the general solution to Equation 86 is given by

$$\boxed{y(x) = c_1e^{r_1x} + c_2e^{r_2x}.}$$

2. $b^2 = 4ac$. In this case we know that $r_1 = r_2$, and that r_1 is real. In this case, our method only produces a single solution $y_1(x) = \exp(r_1x)$. We can use the method of reduction of reduction of order to produce a second solution.

3. $b^2 < 4ac$. Then we have that $r_1 \neq r_2$ and that both r_1 and r_2 are complex numbers. Further, we know that r_1 is the *complex conjugate* of r_2. We will later see how to produce real solutions in this case.

15.1 Two distinct real roots

Example 26 We want to solve the following ODE

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$$

The corresponding characteristic equation is

$$r^2 + 5r + 6 = 0$$

we can see that $r^2+5r+6=(r+2)(r+3)$, thus the two roots of the characteristic equation are $r_1=-2$ and $r_2=-3$. The general solution is then given by

$$y(x)=c_1e^{-2x}+c_2e^{-3x}.$$

We now solve an initial value problem given by $y(0)=2$ and $y'(0)=3$. This leads to the following two equations

$$\begin{aligned} c_1+c_2&=2\\ -2c_1-3c_2&=3 \end{aligned}$$

The solutions to such system are $c_1=9$ and $c_2=-7$. This gives us the particular solution to this initial value problem

$$y(x)=9e^{-2x}-7e^{-3x}.$$

Before we plot the solution, we can qualitatively see how it should look like. We know, from the initial conditions, that at zero the value of the solution is 2 and its tangent has a positive slope of 3. Thus we expect the solution to increase near zero. Next, we see that as x goes to ∞, both exponentials go to zero. Thus the solution tends to zero as x increases. Figure 29 is a plot of the particular solution.

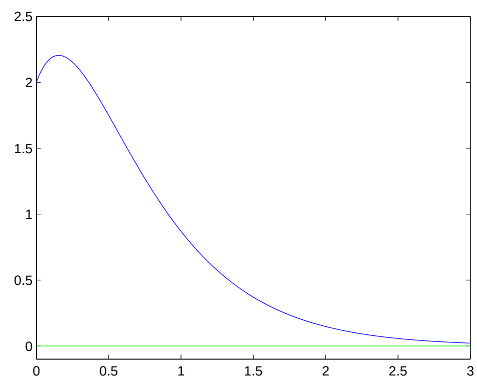


Figure 29: Particular solution

15.2 Double root

This is the case $b^2=4ac$. We know that $r_1=r_2$, and that $r_1=-b/(2a)$ is real. We have only one solution:

$$y_1(x)=e^{-\frac{b}{2a}x}.$$

We can use the method of reduction of order to produce a second solution. We thus know that the formula for a second solution is given by

$$y_2(x)=y_1(x)\int\left(\frac{e^{-\int p(x)dx}}{y_1^2(x)}\right)dx$$

where $p(x) = b/a$ (recall that the equation has to be in standard form for us to apply the reduction of order method). We have that

$$\begin{aligned} y_2(x) &= y_1(x) \int \left(\frac{e^{-\int p(x)dx}}{y_1^2(x)} \right) dx \\ &= e^{-\frac{b}{2a}x} \int \left(\frac{e^{-\int \frac{b}{a}dx}}{\left(e^{-\frac{b}{2a}x}\right)^2} \right) dx \\ &= e^{-\frac{b}{2a}x} \int \left(\frac{e^{-\frac{b}{a}x}}{e^{-2\frac{b}{2a}x}} \right) dx \\ &= e^{-\frac{b}{2a}x} \int \left(\frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}} \right) dx \\ &= e^{-\frac{b}{2a}x} \int 1 dx \\ &= e^{-\frac{b}{2a}x}(x+c) \\ &= xe^{-\frac{b}{2a}x} + ce^{-\frac{b}{2a}x} \end{aligned}$$

but given that we are only looking for another solution, and that we later take a linear combination with $y_1(x)$, we can choose $c = 0$. Thus we get that the second solution for the ODE is

$$y_2(x) = xe^{-\frac{b}{2a}x}$$

which leads us to the general solution

$$\boxed{y(x) = (c_1 + c_2 x)e^{-\frac{b}{2a}x}} \tag{88}$$

Example 27 Consider the equation $y'' - y' + 1/4\ y = 0$. Here we have that $b^2 = 4ac$, thus the only solution to the characteristic equation is given by $r = 1/2$. We use Equation 88 to get the general solution

$$y(x) = (c_1 + c_2 x)e^{\frac{1}{2}x}.$$

We also have the initial conditions $y(0) = 2$ and $y'(0) = 1/3$. This gives the following set of equations

$$\begin{aligned} y(0) &= (c_1 + c_2 0)e^{\frac{1}{2}0} = c_1 = 2 \\ y'(0) &= \left(\frac{1}{2}c_1 + c_2 + \frac{1}{2}c_2 0 \right) e^{\frac{1}{2}0} = \frac{1}{2}c_1 + c_2 = \frac{1}{3} \end{aligned}$$

and we can see that the solution to this system is $c_1 = 2$ and $c_2 = -2/3$. Thus the particular solution is

$$y(x) = \left(2 - \frac{2}{3}x \right) e^{\frac{1}{2}x}.$$

We now study qualitatively how the solution behaves as $x \to +\infty$. The first term is a linear term with negative slope, thus it goes to $-\infty$. The second term is the square root of the exponential function. Thus it goes to $+\infty$. All in all, the solution goes to $-\infty$ as x goes to $+\infty$.

15.3 Complex conjugate roots

This is the case $b^2 < 4ac$. Then we have that $r_1 \neq r_2$ and that both r_1 and r_2 are complex numbers. Further, we know that r_1 is the *complex conjugate* of r_2, which we now denote $\bar{r}_1$. To get an analytical expression for the roots, let us first define the following quantities

$$\lambda = -\frac{b}{2a}, \quad \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

then the roots of the characteristic equation are given by

$$r_1 = \lambda + i\mu \tag{89}$$

$$r_2 = \lambda - i\mu \tag{90}$$

Hence, if we allow the solution to take complex values, we have that the general solution is given by

$$y(x) = c_1 e^{(\lambda + i\mu)x} + c_2 e^{(\lambda - i\mu)x}. \tag{91}$$

We want to find a general solution that does not take complex values. In order to do so, we are going to use Euler's formula (see handout 9.5 as well):

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{92}$$

Thus we have that the general solution is given by

$$\begin{aligned} y(x) &= c_1 e^{(\lambda+i\mu)x} + c_2 e^{(\lambda-i\mu)x} \\ &= c_1 e^{\lambda x} e^{i\mu x} + c_2 e^{\lambda x} e^{-i\mu x} \\ &= c_1 e^{\lambda x}\left[\cos(\mu x) + i\sin(\mu x)\right] + c_2 e^{\lambda x}\left[\cos(\mu x) - i\sin(\mu x)\right] \\ &\stackrel{\text{def}}{=} c_1 \hat{y}_1(x) + c_2 \hat{y}_2(x) \end{aligned}$$

We now define two new solutions **as linear combinations** of the functions $\hat{y}_1(x)$ and $\hat{y}_2(x)$:

$$\begin{aligned} y_1(x) &= \frac{1}{2}\left[\hat{y}_1(x) + \hat{y}_2(x)\right] \\ &= \frac{1}{2}\left[e^{\lambda x}\left[\cos(\mu x) + i\sin(\mu x)\right] + e^{\lambda x}\left[\cos(\mu x) - i\sin(\mu x)\right]\right] \\ &= \frac{1}{2} e^{\lambda x}\left[2\cos(\mu x)\right] = e^{\lambda x}\cos(\mu x) \\ y_2(x) &= \frac{1}{2i}\left[\hat{y}_1(x) - \hat{y}_2(x)\right] \\ &= \frac{1}{2i}\left[e^{\lambda x}\left[\cos(\mu x) + i\sin(\mu x)\right] - e^{\lambda x}\left[\cos(\mu x) - i\sin(\mu x)\right]\right] \\ &= e^{\lambda x}\sin(\mu x) \end{aligned}$$

Given that $\hat{y}_1(x)$ and $\hat{y}_2(x)$ are linearly independent and are solutions of the equation, the general solution to Equation 86 can also be written as

$$\boxed{y(x) = e^{\lambda x}\left[c_1 \cos(\mu x) + c_2 \sin(\mu x)\right]} \tag{93}$$

This is much better. Now we only have functions with real values as expected!

Example 28 Consider the following ODE $16y'' - 8y' + 145y = 0$, subject to $y(0) = -2$ and $y'(0) = 1$. The roots of the characteristic equation are given by $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, with $\lambda = 1/4$ and $\mu = 3$.
The general solution is given by Equation 93. The initial conditions yield the following system of equations

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= \lambda c_1 + \mu c_2. \end{aligned}$$

The solution is given by $c_1 = -2$ and $c_2 = 1/2$. Thus the particular solution is given by

$$y(x) = e^{\frac{1}{4}x}\left[-2\cos(3x) + \frac{1}{2}\sin(3x)\right].$$

From the analysis of the current intensity in an electric circuit under an imposed potential, we know that

$$\left[-2\cos(3x) + \frac{1}{2}\sin(3x)\right]$$

is a sine function with a phase shift and a certain amplitude. We can see a plot of this function in Figure 30.

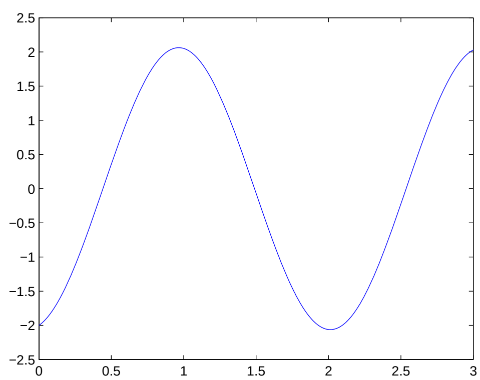


Figure 30: $-2\cos(3x) + \frac{1}{2}\sin(3x)$

The term $\exp(x/4)$ amplifies the previous sine function. We can see in Figure 31 a plot of the solution together with its exponential *envelope*.
The code used to create Figure 31 is the following

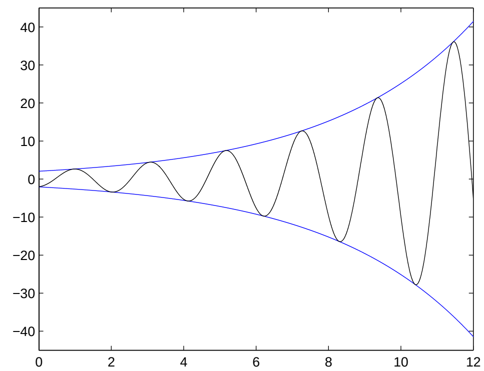


Figure 31: Particular solution $e^{\frac{1}{4}x}\left[-2\cos(3x)+\frac{1}{2}\sin(3x)\right]$ with exponential envelope

```
>> x = 0:.001:20;
>> ys = -2.*cos(3*x)+sin(3*x)./2;
>> m = max(ys);
>> ye = exp(x./4);
>> hold on
>> plot(x,m.*ye);
>> plot(x,-m.*ye);
>> plot(x,ye.*ys,'black');
>> axis([0 12 -45 45]);
```

This is the kind of behavior we expect to find in an oscillating system that is being excited by a driving force. Earthquakes exhibit such behavior.

Example 29 Consider the following ODE $y'' + y' + y = 0$. The roots of the characteristic equation are given by $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, with $\lambda = -1/2$ and $\mu = \sqrt{3}/2$. Thus the general solution is given by

$$y(x) = e^{-\frac{1}{2}x}\left[c_1\cos(\frac{\sqrt{3}}{2}x) + c_2\sin(\frac{\sqrt{3}}{2}x)\right].$$

From the previous example, we know that any such function is a shifted sine function in an exponential envelope. But, as we can see in Figure 32, any solution to this equation decays exponentially fast.
This is the kind of behavior we expect to find in an oscillating system subject to a lot of friction.

Example 30 Consider the following ODE $y'' + 9y = 0$. The roots of the characteristic equation are given by $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, with $\lambda = 0$ and $\mu = 3$. Thus the general solution is given by

$$y(x) = c_1\cos(3x) + c_2\sin(3x).$$

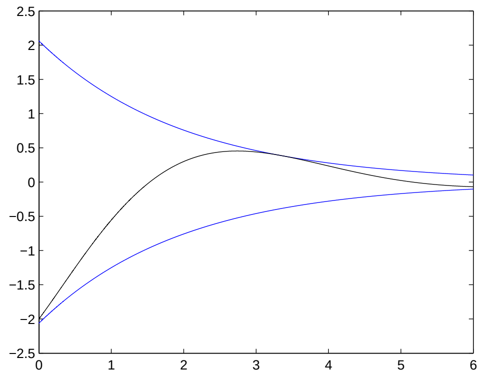


Figure 32: Particular solution $e^{-\frac{1}{2}x}\left[c_1\cos(\frac{\sqrt{3}}{2}x)+c_2\sin(\frac{\sqrt{3}}{2}x)\right]$ with exponential envelope

From the previous example, we know that any such function is a shifted sine function, just like in the pendulum's example and the one in Figure 30.
This is the kind of behavior we expect to find in an oscillating system with no friction.
More generally, from Equation 93 we can see that

- if $\lambda > 0$, then the amplitude of the oscillations of the solution grows exponentially fast, and thus blows up;
- if $\lambda < 0$, then the amplitude of the oscillations of the solution decays exponentially fast, and thus the solution converges to zero;
- if $\lambda = 0$, then the solution is a simple sine function with a delay.

16 Inhomogeneous Equations

Second order ODEs are the simplest models of mechanical phenomena. If we have a particular driving force on the system we are modeling, this leads to inhomogeneous equations. We are going to focus on linear equations only.
Recall the general form for a homogeneous, second order linear ODE, in standard form, is

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0.$$

The general form for an inhomogeneous, second order linear ODE, in standard form, is the following

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x). \tag{94}$$

We cannot solve this equation as in the first order linear ODE case. Nevertheless we can still prove some interesting results on the properties of the general solution to this equation.

Proposition 2. *Assume we have two distinct solutions $y_p^{(1)}(x)$ and $y_p^{(2)}(x)$ to Equation 94. Then the function $y_{homo}(x) = y_p^{(1)}(x) - y_p^{(2)}(x)$ is a solution to the homogeneous equation (Equation 73)*

To prove this proposition, we simply calculate the derivatives of y_{homo} and form the expression $y''_{\text{homo}} + p(x)y'_{\text{homo}} + q(x)y_{\text{homo}}$:

$$\begin{aligned}
&= \frac{d^2y_{\text{p}}^{(1)}}{dx^2} - \frac{d^2y_{\text{p}}^{(2)}}{dx^2} + p(x)\frac{dy_{\text{p}}^{(1)}}{dx} - p(x)\frac{dy_{\text{p}}^{(2)}}{dx} + q(x)y_{\text{p}}^{(1)} - q(x)y_{\text{p}}^{(2)} \\
&= \frac{d^2y_{\text{p}}^{(1)}}{dx^2} + p(x)\frac{dy_{\text{p}}^{(1)}}{dx} + q(x)y_{\text{p}}^{(1)} - \frac{d^2y_{\text{p}}^{(2)}}{dx^2} - p(x)\frac{dy_{\text{p}}^{(2)}}{dx} - q(x)y_{\text{p}}^{(2)} \\
&= \frac{d^2y_{\text{p}}^{(1)}}{dx^2} + p(x)\frac{dy_{\text{p}}^{(1)}}{dx} + q(x)y_{\text{p}}^{(1)} - \left(\frac{d^2y_{\text{p}}^{(2)}}{dx^2} + p(x)\frac{dy_{\text{p}}^{(2)}}{dx} + q(x)y_{\text{p}}^{(2)}\right) \\
&= r(x) - r(x) \\
&= 0.
\end{aligned}$$

Let the general solution to the homogeneous Equation 73 be $c_1y_1(x) + c_2y_2(x)$, and assume that $y_{\text{p}}^{(1)}(x)$ and $y_{\text{p}}^{(2)}(x)$ are two distinct solutions to Equation 94. From Proposition 2, we know that there are constants c'_1 and c'_2 such that $y_{\text{p}}^{(1)}(x) - y_{\text{p}}^{(2)}(x) = c'_1y_1(x) + c'_2y_2(x)$. Thus we just proved following important result.

Theorem 2. *Assume that $y_h(x) = c_1y_1(x) + c_2y_2(x)$ is the general solution to the homogeneous Equation 73 and that $y_p(x)$ is a solution to the inhomogeneous Equation 94. Then the general solution to Equation 94 is given by*

$$y(x) = y_h(x) + y_p(x). \tag{95}$$

We call y_p a particular solution *to the inhomogeneous ODE.*

Hence we have a strategy to find the general solution to Equation 94:

- Solve, if possible, the homogeneous Equation 73. Let $y_{\text{h}}(x) = c_1y_1(x) + c_2y_2(x)$ be the general solution.
- Find a particular solution $y_{\text{p}}(x)$ to the inhomogeneous Equation 94.
- The general solution is then given by Equation 95: $y(x) = y_{\text{h}}(x) + y_{\text{p}}(x)$

17 Undetermined Coefficients Technique

Given that, for second order linear homogeneous ODEs with constant coefficients we can always find the general solution, we now focus on techniques to find a particular solution y_{p} to Equation 94. We are going to explore two techniques to find particular solutions to Equation 94.

The first technique, called the *undetermined coefficients* technique is only applicable when the right hand side, $r(t)$, has a particular form.
The second technique, called the *variation of parameters* technique always give an answer, but is more complicated to apply.
We provide another handout with a summary of the undetermined coefficients technique. There are three rules to using this technique.

- **Rule #1:** Try finding a particular solution that *resembles* the right hand side $r(t)$.

  Example 31

  We are interested in finding the general solution to the ODE

  $$y'' - 3y' - 4y = 3e^{2t}. \tag{96}$$

  We begin by finding the general solution to the homogeneous equation

  $$y'' - 3y' - 4y = 0. \tag{97}$$

  The characteristic equation $r^2 - 3r - 4 = 0$ has $r_1 = 4$ and $r_2 = -1$ as solutions. Thus the general solution to Equation 97 is given by

  $$y_{\mathrm{h}}(t) = c_1e^{4t} + c_2e^{-t}. \tag{98}$$

  In order to find a particular solution to Equation 96, the fundamental idea used in the undetermined coefficients technique is to assume that the particular solution *resembles* the right hand side $r(t)$. In this case, we look for a particular solution of the form $y_{\mathrm{p}}(t) = A\exp(2t)$.

  $$\begin{aligned} y_p'' - 3y_p' - 4y_{\mathrm{p}} &= (2^2)Ae^{2t} - 3(2)Ae^{2t} - 4Ae^{2t} \\ &= -6Ae^{2t} \end{aligned}$$

  thus, if we want y_{p} to be a solution of Equation 96, we need

  $$y_p'' - 3y_p' - 4y_{\mathrm{p}} = 3e^{2t}$$

  which is equivalent to $-6A = 3$, or $A = -1/2$. Thus a particular solution is given by

  $$y_{\mathrm{p}}(t) = -\frac{1}{2}e^{2t}$$

  and the general solution is given by

  $$y(t) = y_{\mathrm{h}}(t) + y_{\mathrm{p}}(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}.$$

There are some problems when applying the technique of undetermined coefficients as such. The following example illustrates one such problem.

Example 32

We are interested in finding the general solution to the ODE

$$y'' - 3y' - 4y = 2\sin(t). \tag{99}$$

We see that the associated homogeneous equation is Equation 97, thus we only need to find a particular solution y_p. We want to look for a particular solution that resembles the right hand side $2\sin(t)$ of Equation 99. But sine and cosine functions are different from exponentials in that the derivative of sine is cosine and that of cosine is minus sine. Thus we look for a particular solution of the form $y_\text{p}(t) = A\sin(t) + B\cos(t)$.

$$\begin{aligned} y_p'' - 3y_p' - 4y_\text{p} &= [-A\sin(t) - B\cos(t)] - 3\,[A\cos(t) - B\sin(t)] - 4\,[A\sin(t) + B\cos(t)] \\ &= [-A + 3B - 4A]\sin(t) + [-B - 3A - 4B]\cos(t) \\ &= [-5A + 3B]\sin(t) + [-3A - 5B]\cos(t) \end{aligned}$$

thus, if we want y_p to be a solution of Equation 99, we need

$$y_p'' - 3y_p' - 4y_\text{p} = 2\sin(t)$$

which is equivalent to the system of equations

$$\begin{aligned} -5A + 3B &= 2 \\ -3A - 5B &= 0. \end{aligned}$$

The solution to such system is $A = -5/17$ and $B = 3/17$. Thus a particular solution is given by

$$y_\text{p}(t) = -\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t)$$

and the general solution is given by

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t).$$

The previous example illustrates the fact that if the right hand side is a sine or cosine function, we need to look for a particular solution as a linear combination of sine and cosine functions.

The next example illustrates how to find a particular solution when the right hand side is a polynomial function.

Example 33

We are interested in finding the general solution to the ODE

$$y'' - 3y' - 4y = 4t^2 - 1. \tag{100}$$

We see that the associated homogeneous equation is Equation 97, thus we only need to find a particular solution y_p. We want to look for a particular solution that resembles the right hand side $4t^2 - 1$ of Equation 100. We look for a particular solution of the form $y_\text{p}(t) = At^2 + Bt + C$.

$$\begin{aligned} y_p'' - 3y_p' - 4y_\text{p} &= [2A] - 3\,[2At + B] - 4\left[At^2 + Bt + C\right] \\ &= [-4A]\,t^2 + [-6A - 4B]\,t + [2A - 3B - 4C] \end{aligned}$$

thus, if we want y_p to be a solution of Equation 100, we need

$$y_p'' - 3y_p' - 4y_\text{p} = 4t^2 - 1$$

which is equivalent to the system of equations

$$\begin{aligned} -4A &= 4 \\ -6A - 4B &= 0 \\ 2A - 3B - 4C &= -1. \end{aligned}$$

The solution to such system is $A = -1$, $B = 3/2$ and $C = -11/8$. Thus a particular solution is given by

$$y_\text{p}(t) = -t^2 + \frac{3}{2}t - \frac{11}{8}$$

and the general solution is given by

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - t^2 + \frac{3}{2}t - \frac{11}{8}.$$

The next example illustrates how to find a particular solution when the right hand side is a product of an exponential function and either a circular function or a polynomial function.

Example 34

We are interested in finding the general solution to the ODE

$$y'' - 3y' - 4y = -8e^t \cos(2t). \tag{101}$$

We see that the associated homogeneous equation is Equation 97, thus we only need to find a particular solution y_p. We want to look for a particular solution that resembles the right hand side $-8e^t\cos(2t)$ of Equation 101. We look for a particular solution of the form $y_{\mathrm{p}}(t) = \exp(t)\,[A\cos(2t) + B\sin(2t)]$. Let's first calculate the derivatives of y_{p}.

$$\begin{aligned} y_p'(t) =& e^t\,[A\cos(2t) + B\sin(2t)] + e^t\,[-2A\sin(2t) + 2B\cos(2t)] \\ =& e^t\,[(A+2B)\cos(2t) + (-2A+B)\sin(2t)] \\ y_p''(t) =& e^t\,[(A+2B)\cos(2t) + (-2A+B)\sin(2t)] \\ &+ e^t\,[-2(A+2B)\sin(2t) + 2(-2A+B)\cos(2t)] \\ =& e^t\,[(A+2B+2B-4A)\cos(2t) + (-2A+B-2A-4B)\sin(2t)] \\ =& e^t\,[(-3A+4B)\cos(2t) + (-4A-3B)\sin(2t)]\,. \end{aligned}$$

Thus we have that

$$\begin{aligned} y_p'' - 3y_p' - 4y_{\mathrm{p}} =& e^t\,[(-3A+4B)\cos(2t) + (-4A-3B)\sin(2t)] \\ &- 3e^t\,[(A+2B)\cos(2t) + (-2A+B)\sin(2t)] \\ &- 4e^t\,[A\cos(2t) + B\sin(2t)] \\ =& e^t\,[(-3A+4B-3A-6B-4A)\cos(2t)] \\ &+ e^t\,[(-4A-3B+6A-3B-4B)\sin(2t)] \\ =& e^t\,[(-10A-2B)\cos(2t) + (2A-10B)\sin(2t)] \end{aligned}$$

thus, if we want y_{p} to be a solution of Equation 100, we need

$$y_p'' - 3y_p' - 4y_{\mathrm{p}} = -8e^t\cos(2t)$$

which is equivalent to the system of equations

$$\begin{aligned} -10A - 2B &= -8 \\ 2A - 10B &= 0. \end{aligned}$$

The solution to such system is $A = 10/13$ and $B = 2/13$. Thus a particular solution is given by

$$y_{\mathrm{p}}(t) = e^t\left[\frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t)\right]$$

and the general solution is given by

$$y(t) = c_1e^{4t} + c_2e^{-t} + e^t\left[\frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t)\right].$$

- **Rule #2:** If the right hand side is a sum of several terms, take each term individually and then add all the particular solutions together. More formally, if $r(t) = r_1(t) + r_2(t)$, then

  1. find $y_{\mathrm{p}}^{(1)}(t)$, a particular solution to the ODE with $r_1(t)$ as its right hand side;

2. find $y_{\mathrm{p}}^{(2)}(t)$, a particular solution to the ODE with $r_2(t)$ as its right hand side;
3. finally, set $y_{\mathrm{p}}(t) = y_{\mathrm{p}}^{(1)}(t) + y_{\mathrm{p}}^{(2)}(t)$.

Example 35

We are interested in finding the general solution to the ODE

$$y'' - 3y' - 4y = 3e^{2t} - 8e^{t}\cos(2t). \tag{102}$$

1. We begin by finding a particular solution to $y'' - 3y' - 4y = 3e^{2t}$. But this is Equation 96, thus
$$y_{\mathrm{p}}^{(1)} = -\frac{1}{2}e^{2t}.$$
2. We then find a particular solution to $y'' - 3y' - 4y = -8e^{t}\cos(2t)$. But this is Equation 101, thus
$$y_{\mathrm{p}}^{(2)}(t) = e^{t}\left[\frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t)\right].$$

Now we have that a particular solution is

$$y_{\mathrm{p}}(t) = y_{\mathrm{p}}^{(1)} + y_{\mathrm{p}}^{(2)} = -\frac{1}{2}e^{2t} + e^{t}\left[\frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t)\right]$$

which leads to the general solution to Equation 102

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t} + e^{t}\left[\frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t)\right].$$

- **Rule #3:** This is a more complicated rule. Let us begin with an example.

Example 36

We are interested in finding the general solution to the ODE

$$y'' + 4y = 3\cos(2t). \tag{103}$$

Let us try to find a particular solution of the form $y_{\mathrm{p}}(t) = A\cos(2t) + B\sin(2t)$.

$$\begin{aligned} y_p'' + 4y_{\mathrm{p}} &= [-4A\cos(2t) - 4B\sin(2t)] + 4\,[A\cos(2t) + B\sin(2t)] \\ &= [-4A + 4A]\cos(2t) + [-4B + 4B]\sin(2t) \\ &= 0. \end{aligned}$$

Thus there it is impossible to find A and B such that the particular solution is of the form $y_{\mathrm{p}}(t) = A\cos(2t) + B\sin(2t)$.

After further inspection, this result is not surprising. If we consider the homogeneous equation $y'' + 4y = 0$, its general solution is given by $y_{\mathrm{h}}(t) = c_1\cos(2t) + c_2\sin(2t)$, which is the same form we were trying to use for the particular solution! Thus, in this case the right hand side $3\cos(2t)$ is a solution to the homogeneous equation.

Rule #3 stipulates that, if the right hand side $r(t)$ is a solution to the homogeneous equation, then assume that the particular solution is of the form

$$y_{\mathrm{p}}(t) = t\ y_{\mathrm{h}}(t).$$

We go back to our example, and look for a particular solution of the form

$$y_{\mathrm{p}}(t) = t\left(A\cos(2t) + B\sin(2t)\right).$$

Let us first calculate its derivatives.

$$\begin{aligned}
y_p'(t) =& \left(A\cos(2t) + B\sin(2t)\right) + t\left(-2A\sin(2t) + 2B\cos(2t)\right)\\
y_p''(t) =& \left(-2A\sin(2t) + 2B\cos(2t)\right) + \left(-2A\sin(2t) + 2B\cos(2t)\right)\\
&+ t\left(-4A\cos(2t) - 4B\sin(2t)\right)\\
=& 4\left(-A\sin(2t) + B\cos(2t)\right) - 4y_{\mathrm{p}}(t).
\end{aligned}$$

Thus we have that

$$\begin{aligned}
y_p''(t) + 4y_{\mathrm{p}}(t) &= 4\left(-A\sin(2t) + B\cos(2t)\right) - 4y_{\mathrm{p}}(t) + 4y_{\mathrm{p}}(t)\\
&= -4A\sin(2t) + 4B\cos(2t).
\end{aligned}$$

In order for y_{p} to be a solution to Equation 103, we need that

$$\begin{aligned}
y_p''(t) + 4y_{\mathrm{p}}(t) &= 3\cos(2t), \quad \text{which leads to}\\
-4A\sin(2t) + 4B\cos(2t) &= 3\cos(2t).
\end{aligned}$$

Thus we have that $A = 0$ and $B = 3/4$. Hence the particular solution to Equation 103 is

$$y_{\mathrm{p}}(t) = \frac{3}{4}t\sin(2t)$$

and the general solution is

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{3}{4}t\sin(2t).$$

Note: Recall that rule #3 stipulates that, in the event that the right hand side $r(t)$ is a solution to the homogeneous equation, we should look for a particular solution of the form $y_{\mathrm{p}}(t) = ty_{\mathrm{h}}(t)$. However, if the characteristic equation of 73 has only one solution $r_1 = \frac{-b}{2a}$, then the general solution to Equation 73 is given by

$$y_{\mathrm{h}}(t) = e^{-\frac{bt}{2a}}\left(c_1 + c_2 t\right).$$

A multiplication by t is no longer sufficient. We need to multiply by t^2:

$$y_\mathrm{p}(t) = At^2 e^{-\frac{tb}{2a}}.$$

You never have to go beyond multiplying either by t or t^2.

18 Variation of Parameters

Recall that we are considering second order, linear and inhomogeneous ODEs. If we write them in standard form, we have

$$y'' + p(t)y' + q(t)y = r(t).$$

We associate the following second order, linear and homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0.$$

We proved that, if $y_\mathrm{h} = c_1 y_1 + c_2 y_2$ is the general solution to Equation 73 and y_p is a particular solution to 94, then the general solution to Equation 94 is given by

$$y(t) = y_\mathrm{h}(t) + y_\mathrm{p}(t).$$

Thus, one can devise the following strategy to find the general solution to Equation 94.

- *Step 1*: Find the general solution y_h to the homogeneous Equation 73.
- *Step 2*: Find *a* particular solution y_p to Equation 94.

As mentioned before, finding the general solution to the homogeneous equation is, in general, a difficult problem. Nevertheless, when the functions $p(t)$ and $q(t)$ are constants, we provided a technique to calculate the general solution to the homogeneous equation.
In that setting, we presented the method of undetermined coefficients to calculate a particular solution y_p to Equation 94 when the function $r(t)$ had a particular form. But we are not yet able to solve the inhomogeneous equation for *all* possible functions $r(t)$. The method of variation of parameters allows us to do just that:

Theorem 3 (Variation of Parameters)**.** *Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation. Then y_p, solution to the inhomogeneous Equation 94, can be obtained using:*

$$y_p(t) = -y_1(t)\int \frac{r\, y_2}{W}dt + y_2(t)\int \frac{r\, y_1}{W}dt, \quad \textit{where} \tag{104}$$

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t). \tag{105}$$

Proof: The idea behind this method is a generalization of the method we used to find the general solution to first order, linear, inhomogeneous ODEs. We look for a solution y_{p} of the form

$$y_{\text{p}}(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{106}$$

where u_1 and u_2 are function we want to determine.
Given that y_{p} is a solution to Equation 94, we will get a relation between u_1, u_2 and their derivatives by replacing y_{p} in Equation 94 with the form given by Equation 106.
Let us first calculate the first derivative of y_{p}:

$$\begin{aligned} y_{\text{p}}' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\ &= u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2' \end{aligned}$$

In order to simplify this equation we are going to require that:

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \tag{107}$$

With this equation, we get:

$$y_{\text{p}}' = u_1y_1' + u_2y_2'.$$

We now calculate the second derivative of y_{p}:

$$y_{\text{p}}'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

Thus we have that

$$\begin{aligned} y_{\text{p}}'' + p(t)y_{\text{p}}' + q(t)y_{\text{p}} &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \\ &\quad + p(t)\left(u_1y_1' + u_2y_2'\right) \\ &\quad + q(t)\left(u_1y_1 + u_2y_2\right) \\ &= u_1\left(y_1'' + p(t)y_1' + q(t)y_1\right) \\ &\quad + u_2\left(y_2'' + p(t)y_2' + q(t)y_2\right) \\ &\quad + u_1'y_1' + u_2'y_2' \\ &= u_1'y_1' + u_2'y_2'. \end{aligned}$$

because y_1 and y_2 are solutions of the homogeneous equation. But y_{p} is a solution to Equation 94, thus we have that

$$u_1'y_1' + u_2'y_2' = r(t). \tag{108}$$

As a summary, we have that u_1' and u_2' satisfy simultaneously Equations 107 and 108

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= r(t). \end{aligned}$$

This is simply of system of 2 linear equations in u_1' and u_2'. This system can be solved to obtain:

$$u_1' = -\frac{r(t)y_2}{y_1y_2' - y_1'y_2}, \quad u_2' = \frac{r(t)y_1}{y_1y_2' - y_1'y_2}.$$

We have obtained a particular solution to Equation 94 in the form:

$$\begin{aligned} y_{\mathrm{p}}(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= -y_1(t)\int \frac{r(t)\,y_2}{y_1y_2' - y_1'y_2}dt + y_2(t)\int \frac{r(t)\,y_1}{y_1y_2' - y_1'y_2}dt, \end{aligned}$$

This is the desired result. ■

Example 37

Consider the following equation

$$y'' + y = \frac{1}{\cos(t)}.$$

The solutions y_1 and y_2 to the homogeneous equation are given by

$$\begin{aligned} y_1(t) &= \cos(t) \\ y_2(t) &= \sin(t). \end{aligned}$$

Thus we have that

$$\begin{aligned} W(t) &= y_1(t)y_2'(t) - y_1'(t)y_2(t) \\ &= \cos(t)\cos(t) - (-\sin(t))\sin(t) \\ &= \cos^2(t) + \sin^2(t) \\ &= 1. \end{aligned}$$

Hence, from Equation 104 we have a particular solution:

$$\begin{aligned} y_{\mathrm{p}}(t) &= -y_1(t)\int \frac{ry_2}{W}dt + y_2(t)\int \frac{ry_1}{W}dt, \quad \text{where} \\ &= -\cos(t)\int \frac{\frac{1}{\cos(t)}\sin(t)}{1}dt + \sin(t)\int \frac{\frac{1}{\cos(t)}\cos(t)}{1}dt \\ &= -\cos(t)\int \frac{\sin(t)}{\cos(t)}dt + \sin(t)\int 1\,dt \\ &= \cos(t)\ln|\cos(t)| + t\sin(t). \end{aligned}$$

Thus the general solution is given by

$$y(t) = c_1\cos(t) + c_2\sin(t) + \cos(t)\ln|\cos(t)| + t\sin(t).$$

Example 38

Consider the following equation

$$y'' + y = \tan(t).$$

Given that the homogeneous equation is identical to that of Example 37, we have that

$$\begin{aligned} y_1(t) &= \cos(t) \\ y_2(t) &= \sin(t). \end{aligned}$$

Hence, from Equation 104 we have that a particular solution to our ODE is given by

$$\begin{aligned} y_{\mathrm{p}}(t) &= -y_1(t) \int \frac{ry_2}{W} dt + y_2(t) \int \frac{ry_1}{W} dt, \quad \text{where} \\ &= -\cos(t) \int \frac{\tan(t)\sin(t)}{1} dt + \sin(t) \int \frac{\tan(t)\cos(t)}{1} dt \\ &= -\cos(t) \int \frac{\sin^2(t)}{\cos(t)} dt + \sin(t) \int \sin(t) dt \\ &= -\cos(t) \int \frac{1-\cos^2(t)}{\cos(t)} dt + \sin(t)(-\cos(t)) \\ &= -\cos(t) \left[\int \frac{1}{\cos(t)} dt - \int \cos(t) dt \right] - \sin(t)\cos(t) \\ &= -\cos(t) \left[\ln\left| \frac{1}{\cos(t)} + \tan(t) \right| - \sin(t) \right] - \sin(t)\cos(t) \\ &= -\cos(t) \ln\left| \frac{1}{\cos(t)} + \tan(t) \right|. \end{aligned}$$

Thus the general solution is given by

$$y(t) = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln\left| \frac{1}{\cos(t)} + \tan(t) \right|.$$

The following example illustrates the use of all of the methods we discussed so far.

Exercise.
Consider the following ODE

$$y'' - ty' + y = e^t.$$

Here, the right hand side $r(t) = \exp(t)$ is a simple exponential function. If we try to apply the method of undetermined coefficients, we would look for a particular solution y_{p} of the form $y_{\mathrm{p}} = A\exp(t)$:

$$y_{\mathrm{p}}'' - ty_{\mathrm{p}}' + y_{\mathrm{p}} = Ae^t - Ate^t + Ae^t = (2-t)Ae^t$$

Thus, for y_{p} to be a particular solution, we would need to find A such that, *for all t*

$$(2-t)Ae^t = e^t$$

which yields $(2-t)A = 1$, and we conclude that there is no such constant A.
The reason why the method of undetermined coefficients failed is that the proposed ODE does not have constant coefficients.
In order to provide the general solution to this ODE, we first note that $y_1(t) = t$ is a solution to the homogeneous ODE $y'' - ty' + y = 0$. Thus we can use the method of reduction of order to get a second solution $y_2(t)$ to the homogeneous equation, and thus its general solution $y_\text{h} = c_1 y_1 + c_2 y_2$. Next we can use the method of variation of parameters to get a particular solution y_p to the ODE. Finally, the general solution is given by Equation 95.

Summary. We now summarize our findings. In order to solve a linear, second order, inhomogeneous ODE, given by Equation 94, we proceed as follows.

- *Step 1*: Solve the homogeneous Equation 73. To do so, we consider two cases:
  - If the equation has constant coefficients, this is simple.
  - If the equation does not have constant coefficients, we
    1. find a solution y_1 by "inspection" (or by guessing/being lucky)
    2. find a second solution using the method of reduction of order.
- *Step 2*: Find a particular solution to the in-homogeneous Equation 94. To do so, we consider two cases:
  - If the equation has constant coefficients *and* the right hand side $r(t)$ has a simple form, then we use the method of undetermined coefficients (see the handout for specifics on the form of the right hand side $r(t)$).
  - In all other cases, use the method of variation of parameters.

19 Mass Spring Systems

Modeling mass spring systems is a natural application of second order ODE theory. From a mathematical point of view, mass spring systems are equivalent to RLC-circuits. Thus mass spring systems will exhibit the same type of behavior as RLC-circuits.
The system considered is represented in Figure 33. The origin of the x axis is chosen to correspond to the mass at equilibrium.
From the definition of x's axis origin, we deduce that the gravitational force $m\vec{g}$ is equal to the opposite of the spring force at equilibrium. Thus, the only forces to consider in this mass spring system are

- the spring force; and
- friction.

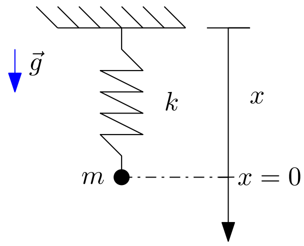


Figure 33: Mass Spring System

Given the definition of the axis' origin, we deduce that the spring force, when projected along the x axis, is equal to $-kx$. We will introduce a model for friction when we study *damped* mass spring systems.

19.1 Undamped Motion

In this subsection we neglect friction, thus the second law of motion dynamics yields

$$m\frac{d^2x}{dt^2} = -kx$$

which we rewrite as

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \quad \text{where} \tag{109}$$

$$\omega^2 = \frac{k}{m} \tag{110}$$

The characteristic equation associated to Equation 109 is

$$r^2 + \omega^2 = 0$$

and its solutions are given by $r = \pm i\omega$. Thus the motion of an undamped mass spring system is given by

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t). \tag{111}$$

This is a periodic motion. The period T and frequency f of the motion are given by

$$T = \frac{2\pi}{\omega} \tag{112}$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{113}$$

where the period is given in seconds (s), and the frequency is given in s^{-1}, or Hertz (Hz). The frequency is the number of oscillations per second. Figure 34 illustrates how to measure a period from a graph.

Example 39

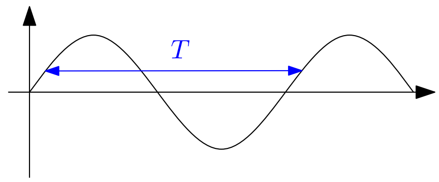


Figure 34: Period T of a periodic function

Assume that $m = 1/16\text{kg}$ and $k = 4\text{kg/s}^2$. Then we have that $\omega^2 = k/m = 64$. The equation for the general motion of this particular mass spring system is thus given by

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t).$$

Now, if we assume that we are given the following initial conditions

$$x(0) = \frac{2}{3}$$
$$x'(0) = -\frac{4}{3}$$

then we have that

$$x(0) = c_1 \cos(8 * 0) + c_2 \sin(8 * 0) = c_1 = \frac{2}{3}$$

and, from

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

we get

$$x'(0) = -8c_1 \sin(8 * 0) + 8c_2 \cos(8 * 0) = 8c_2 = -\frac{4}{3}$$

or $c_2 = -1/6$. Thus the solution to the initial value problem is

$$x(t) = \frac{2}{3} \cos(8t) - \frac{1}{6} \sin(8t). \tag{114}$$

From Figure 35, we can see that the resulting function is a shifted sine function.

Computing the amplitude and delay for Sine and Cosine Functions

We now concentrate on providing a general method to transform a linear combination of sine and cosine functions into a scaled and shifted sine function. Let $x(t)$ be a linear combination of $\sin(\omega t)$ and $\cos(\omega t)$.

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t). \tag{115}$$

We want to find $A > 0$ and ϕ such that

$$x(t) = A \sin(\omega t + \phi). \tag{116}$$

Let us expand Equation 116:

$$A \sin(\omega t + \phi) = A \sin(\omega t) \cos(\phi) + A \cos(\omega t) \sin(\phi).$$

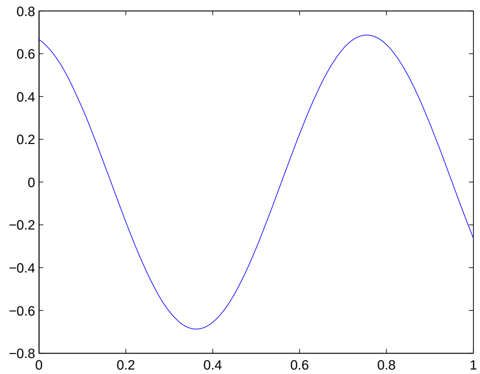


Figure 35: Solution defined by Equation 114

Thus we see that ϕ is implicitly defined by

$$\cos(\phi) = \frac{c_2}{A} \tag{117}$$

$$\sin(\phi) = \frac{c_1}{A}. \tag{118}$$

Depending on the sign of $\cos(\phi)$ and $\sin(\phi)$, we can deduce boundaries on the value of ϕ. In Figure 36 we see that

- if both are positive, ϕ is in the first quadrant;
- if $\cos(\phi) < 0$ and $\sin(\phi) > 0$, ϕ is in the second quadrant;
- if both are negative, ϕ is in the third quadrant;
- if $\cos(\phi) > 0$ and $\sin(\phi) < 0$, ϕ is in the fourth quadrant.

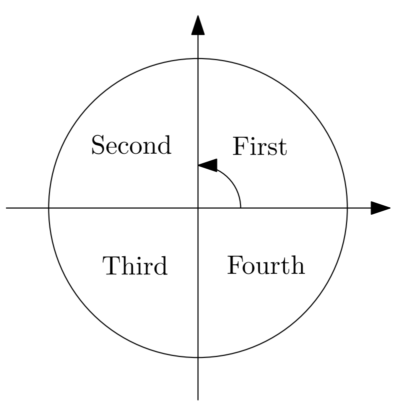


Figure 36: Quadrants for boundaries on the value of ϕ

We can prove (using $\cos^2 + \sin^2 = 1$) that A is equal to:

$$A = \sqrt{c_1^2 + c_2^2}$$

and ϕ must satisfy:

$$\tan\phi = \frac{c_1}{c_2}$$

With the understanding above regarding which quadrant ϕ is in, this completely characterizes ϕ.

We now return to Example 39. Recall that the solution to the initial value problem is given by Equation 114:

$$x(t) = \frac{2}{3}\cos(8t) - \frac{1}{6}\sin(8t).$$

In this case:

$$\begin{aligned} A &= \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \frac{\sqrt{17}}{6} \approx 0.7\text{m} \\ \cos(\phi) &= \frac{-\frac{1}{6}}{A} = -\frac{1}{\sqrt{17}} \\ \sin(\phi) &= \frac{\frac{2}{3}}{A} = \frac{4}{\sqrt{17}} \end{aligned}$$

Thus we can see that ϕ is in the second quadrant and

$$\phi = \text{atan}(-4) \approx 104 \text{ deg} = 1.816 \text{ rad}$$

Thus we can write the solution as

$$x(t) = \frac{\sqrt{17}}{6}\sin(8t + 1.816).$$

19.2 Free Damped Motion

We now consider friction. Intuitively, all solutions should converge to zero as t goes to infinity. Nevertheless, we have two types of behavior

- oscillatory behavior that converges to zero; and
- exponential decay to zero.

When projected to the x axis, the friction force can be expressed as $-\beta x'(t)$, where $\beta > 0$ is a constant. The friction force is sometimes called the drag force. In order to see that $-\beta x'(t)$ is a reasonable expression for the friction force, let us think about how friction manifests. When a body is at rest it experiences no friction force. When it is in motion, friction force tends to *slow it down*. This explains the sign in the projection. Further, as the speed of the body increases, the friction force increases as well. Thus $-\beta x'(t)$ is the simplest model encompassing the previous two phenomena.

Thus we have that the second law of dynamics, when projected into the x axis, yields

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}, \quad \text{or}$$
$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0.$$

We define

$$\omega^2 = \frac{k}{m}, \quad \text{and}$$
$$2\lambda = \frac{\beta}{m}$$

and get the following general equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \tag{119}$$

with $\lambda > 0$.
Writing the equation in that way is very important. There is a field called "dimensionality analysis", which basically allows selecting the minimum number of parameters that drive the system's dynamics. In our case, we started with a three parameter system (m, k and β), but the dynamics really only depends on two, λ and ω.
In our case, it might seem that dimensionality analysis does not simplify the analysis that much. But it is an important concept in science and engineering. For instance, in fluid mechanics, even though flows are very complicated, one can characterize some of its properties with a single number, called the *Reynolds number*. Thus, say we need to design a wing for an airplane. Building a full size model might be very expensive. Instead, the solution is to choose a model scale and wind speed parameters such that the model and the real system have the same Reynolds number. In this way we can study the physics of a full size airplane wing using a much smaller model.
The roots of the characteristic equation associated with Equation 119 are

$$r = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Hence we will have three cases.

1. Overdamped System: $\lambda > \omega$.
   In this case we get two real roots, and the general solution to Equation 119 is given by

   $$x(t) = c_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}.$$

   Now, it is easy to see that $-\lambda - \sqrt{\lambda^2 - \omega^2} < 0$. Noting that, in a second order polynomial equation of the form $y^2 + ay + c = 0$, the product of the roots is equal to c, we have that

   $$\left(-\lambda - \sqrt{\lambda^2 - \omega^2}\right)\left(-\lambda + \sqrt{\lambda^2 - \omega^2}\right) = \omega^2 > 0,$$

we deduce that $-\lambda - \sqrt{\lambda^2 - \omega^2}$ and $-\lambda + \sqrt{\lambda^2 - \omega^2}$ have the same sign.
Thus $-\lambda + \sqrt{\lambda^2 - \omega^2} < 0$ and $x(t)$ is the sum of two decaying exponential functions.

In Figure 37 we see that, depending on the initial conditions, we get qualitatively two types of behavior.

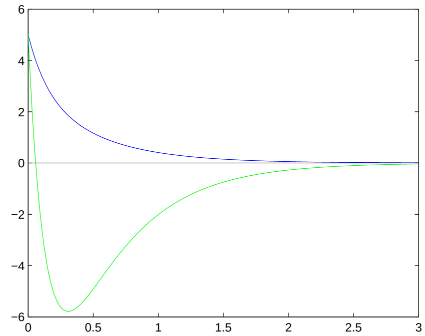


Figure 37: Overdamped system

2. Critical system: $\lambda = \omega$.
   In this case we have a double root $-\lambda$ for the characteristic equation. Thus we get the general solution to Equation 119

$$x(t) = e^{-\lambda t}(c_1 + c_2 t).$$

   This is a very interesting case. As we can see in Figure 38, the solution is zero at most once. Thus we do not get any oscillations, and the friction force is as small as possible.

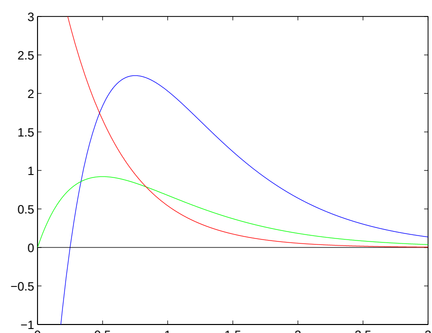


Figure 38: Critical system

   For instance, when designing a car we want to avoid oscillations as much as possible. At the same time, we want to have a pleasant drive, thus the damping in the suspension needs to be not too great (or we get a "bumpy" ride). By properly selecting the parameters of the systems to be close to criticality, we achieve a comfortable ride with few oscillations.

3. Underdamped system: $\lambda < \omega$.
In this case we get two complex conjugate roots to the characteristic equation. Thus the general solution is given by

$$x(t) = e^{-\lambda t}\left[c_1 \cos\left(\sqrt{\omega^2 - \lambda^2}\; t\right) + c_2 \sin\left(\sqrt{\omega^2 - \lambda^2}\; t\right)\right].$$

Here we expect to have shifted sinusoidal oscillations in an exponentially decaying envelope. We can see an example of such a system in Figure 39. The equations of the exponential envelope are given by

$$\sqrt{c_1^2 + c_2^2}\, e^{-\lambda t}, \quad \text{and}$$
$$-\sqrt{c_1^2 + c_2^2}\, e^{-\lambda t}.$$

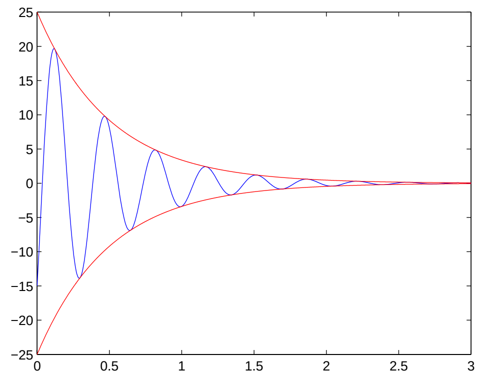


Figure 39: Underdamped system

Example 40

Assume we have a mass spring system with friction with $m = 0.5$, $k = 5$ and $\beta = 1$. Thus we get that the roots of the characteristic equation are $r = -1 \pm 3i$. We are thus in the underdamped case.

We get the following general solution

$$x(t) = e^{-t}\left[c_1 \cos(3t) + c_2 \sin(3t)\right].$$

If we assume the following initial conditions $x(0) = -2$ and $x'(0) = 0$, the solution to the initial value problem is given by

$$x(t) = e^{-t}\left[-2\cos(3t) - \frac{2}{3}\sin(3t)\right].$$

Let us change $-2\cos(3t) + \frac{2}{3}\sin(3t)$ into a scaled and shifted sine function $A\sin(3t+\phi)$:

$$\cos(\phi) = -\frac{2}{3A},\ \sin(\phi) = -\frac{2}{A},\ A = \frac{2}{3}\sqrt{10}.$$

Thus ϕ is in the third quadrant, and we get $\phi = 252$ deg $= 4.4$ rad. We can rewrite the solution to the initial value problem as:

$$x(t) = \frac{2}{3}\sqrt{10}e^{-t}\sin(3t + 4.4).$$

Figure 40 shows a plot of the solution.

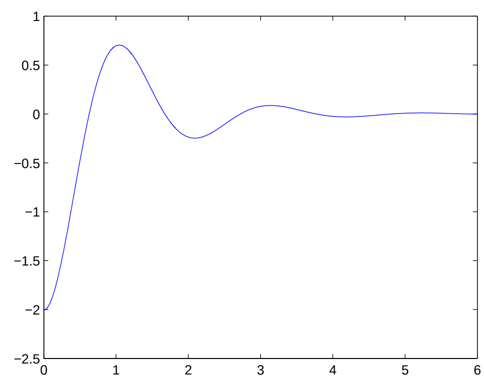


Figure 40: Example of underdamped system

19.3 Forced Oscillations

We are now interested in mechanical systems with a forcing term. The dynamics of a mass-spring system with a forcing term $f(t)$, are described by the following ODE:

$$mx'' = -kx - \beta x' + f(t). \tag{120}$$

But what does the forcing term $f(t)$ correspond to? We can either apply the forcing term to the mass itself, or we can move the support. We can see the later in Figure 41. In the case where the force is applied to the support, we can think of $f(t)$ as, for instance, an earthquake.

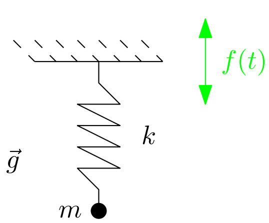


Figure 41: Forced mass-spring system

We can divide Equation 120 by m and get the following equivalent equation

$$x'' + 2\lambda x' + \omega^2 x = F(t), \quad \text{where} \tag{121}$$

$$\lambda = \frac{\beta}{2m}, \quad \text{and} \tag{122}$$

$$\omega^2 = \frac{k}{m}, \quad \text{and} \tag{123}$$

$$F(t) = \frac{f(t)}{m}. \tag{124}$$

The general theory for this type of equation can be found in the textbook. Instead, we will focus on a few examples.

Example 41

Assume that we have $m = 1/5$, $k = 2$ and $\beta = 1.2$. We further assume that the forcing term is $f(t) = 5\cos(4t)$. The equation associated to these parameters is

$$x'' + 6x' + 10x = 25\cos(4t).$$

The associated characteristic equation $r^2 + 6r + 10 = 0$ has two solutions given by $r = -3 \pm i$. Thus the general solution to the homogeneous equation is

$$x_h(t) = e^{-3t}\left[c_1 \cos(t) + c_2 \sin(t)\right].$$

The initial conditions are $x(0) = 1/2$ and $x'(0) = 0$. This is a mass spring system as in Figure 41 where an operator initially pulled on the mass to the position $x = 1/2$, and released the mass with no velocity.

We use the method of undetermined coefficients to look for a particular solution $x_{\text{p}}(t)$. We look for a solution of the form

$$x_{\text{p}}(t) = A\cos(4t) + B\sin(4t).$$

We calculate the first and second derivatives of x_{p} and get

$$\begin{aligned} x'_{\text{p}}(t) &= -4A\sin(4t) + 4B\cos(4t) \\ x''_{\text{p}} &= -16A\cos(4t) - 16B\sin(4t). \end{aligned}$$

Thus we have that x_{p} is a particular solution if

$$\begin{aligned} x''_{\text{p}} + 6x'_{\text{p}} + 10x_{\text{p}} &= -16A\cos(4t) - 16B\sin(4t) \\ &\quad + 6\left[-4A\sin(4t) + 4B\cos(4t)\right] \\ &\quad + 10\left[A\cos(4t) + B\sin(4t)\right] \\ &= \left[-6A + 24B\right]\cos(4t) + \left[-24A - 6B\right]\sin(4t) \\ &= 25\cos(4t). \end{aligned}$$

Thus we get that $-6A+24B = 25$ and $-24A-6B = 0$. The solution is given by $A = -25/102$ and $B = 50/51$. Hence the particular solution is

$$x_{\mathrm{p}}(t) = -\frac{25}{102}\cos(4t) + \frac{50}{51}\sin(4t).$$

Thus the general solution is given by

$$x(t) = e^{-3t}\left[c_1\cos(t) + c_2\sin(t)\right] - \frac{25}{102}\cos(4t) + \frac{50}{51}\sin(4t).$$

From the initial conditions $x(0) = 1/2$ and $x'(0) = 0$, we get

$$c_1 = \frac{38}{51}, \quad c_2 = -\frac{86}{51}.$$

Hence the solution to the initial value problem is given by

$$x(t) = e^{-3t}\left[\frac{38}{51}\cos(t) - \frac{86}{51}\sin(t)\right] - \frac{25}{102}\cos(4t) + \frac{50}{51}\sin(4t). \tag{125}$$

Let us analyze closely what the solution defined by Equation 125 means.

1. It is composed of two parts.
   - The first part, $\exp(-3t)\left[\frac{38}{51}\cos(t) - \frac{86}{51}\sin(t)\right]$, decreases rapidly to zero as t increases. We call it the *transient solution*.
   - The second part, $-\frac{25}{102}\cos(4t) + \frac{50}{51}\sin(4t)$, called the *steady state solution*, is independent of the initial conditions.
2. Second, we note that the initial conditions are used to calculate c_1 and c_2. But both those coefficients are in the transient portion of the solution. Thus, when a driven mass spring system has friction, the behavior of the systems *in the long run* is independent of the initial conditions. Thus we can say that the system "forgets" where it started from.

We can also see that, in the presence of friction, if there is no forcing term, the system loses energy (whose initial quantity is determined by the initial conditions) until the energy is depleted and the system goes to rest.

19.4 Resonance

In order to study the resonance phenomenon, we consider how the behavior of the solution to Equation 121 depends on γ, where the forcing term is of the form $F(t) = \sin(\gamma t)$. We can use the Matlab simulator from file "`MassSpring.m`" to see the change in behavior in real time.

In order to account for the behavior observed using the Matlab simulator, we are going to restrict ourselves to the undamped case. Thus we assume that we have an oscillating system, such as that represented by Figure 41. We assume that the friction force is negligible. We study the response of such system when the forcing term is a sine function:

$$x'' + \omega^2 x = \sin(\gamma t). \tag{126}$$

To further simplify the algebra, we assume that $x(0) = 0$ and $x'(0) = 0$.

19.4.1 Case 1: $\omega \neq \gamma$

In this case we know that $\sin(\gamma t)$ is not a solution to the homogeneous equation. Thus we can use the method of undetermined coefficients and look for a solution of the form $A\sin(\gamma t) + B\cos(\gamma t)$. We can check that

$$x_{\rm p}(t) = \frac{1}{\omega^2 - \gamma^2}\sin(\gamma t)$$

is a particular solution to Equation 126. From the initial condition $x(0) = 0$, we get that the solution to Equation 126 is given by

$$x(t) = A\sin(\omega t) + \frac{1}{\omega^2 - \gamma^2}\sin(\gamma t)$$

From the other initial condition $x'(0) = 0$, we get that the solution to Equation 126 is

$$x(t) = -\frac{\gamma/\omega}{\omega^2 - \gamma^2}\sin(\omega t) + \frac{1}{\omega^2 - \gamma^2}\sin(\gamma t). \tag{127}$$

We now plot the solution from Equation 127 for different values of γ when $\omega = 1$. If $\gamma = 0.1$, we see from Figure 42 that we obtain a slowly oscillating sine wave with some wiggles.

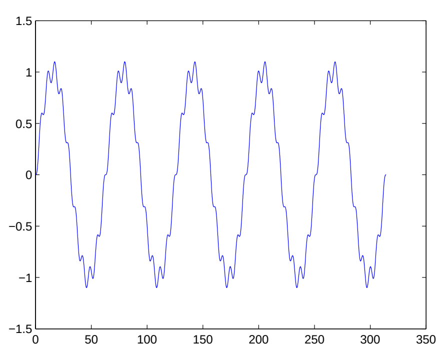


Figure 42: Resonance study $\omega = 1$ and $\gamma = 0.1$

If $\gamma = 1.985$ (γ is closer to ω but still quite different), we see from Figure 43 that the resulting function is now far from a sine wave.

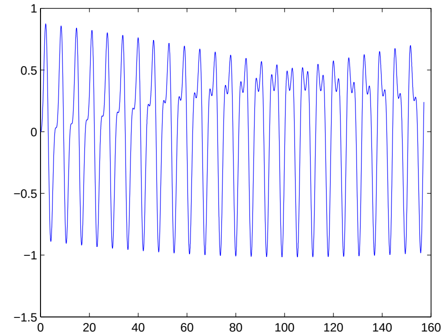


Figure 43: Resonance study $\omega = 1$ and $\gamma = 1.985$

Now if γ is close to ω, say $\gamma = 1.1$, we get an entirely different behavior. The amplitude is much higher. In Figure 44 we observe a sine function oscillating inside a sine envelope.
Finally, if γ is very close to ω, say $\gamma = 1.01$, the amplitude starts growing in an unbounded way. In Figure 45 we see that the amplitude is now around 100. As γ gets closer to ω, the amplitude would keep growing.

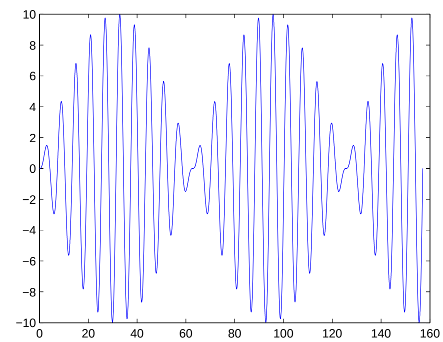


Figure 44: Resonance study $\omega = 1$ and $\gamma = 1.1$

19.4.2 Case 2: $\omega = \gamma$

In this case we know that $\sin(\gamma t)$ is a solution to the homogeneous equation. Thus we can use the method of undetermined coefficients and look for a solution of the form $t(A\sin(\omega t) + B\cos(\omega t))$. We can check that

$$x_{\mathrm{p}}(t) = -\frac{t}{2\omega}\cos(\omega t)$$

is a particular solution to Equation 126. From the initial condition $x(0) = 0$, we get that the solution to Equation 126 is given by

$$x(t) = A\sin(\omega t) - \frac{t}{2\omega}\cos(\omega t)$$

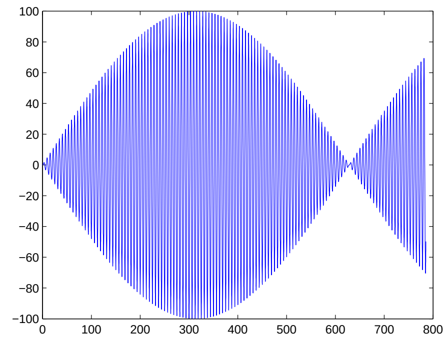


Figure 45: Resonance study $\omega = 1$ and $\gamma = 1.01$

From the other initial condition $x'(0) = 0$, we get that the solution to Equation 126 is

$$x(t) = \frac{1}{2\omega^2}\sin(\omega t) - \frac{t}{2\omega}\cos(\omega t) \tag{128}$$

We can see that the amplitude will grow unbounded because of the term $\frac{t}{2\omega}\cos(\omega t)$. In this case we say that the forced system is in *resonance*. Figure 46 is a plot of the behavior of a system in resonance. We can see that we have a sine function with a linear envelope.

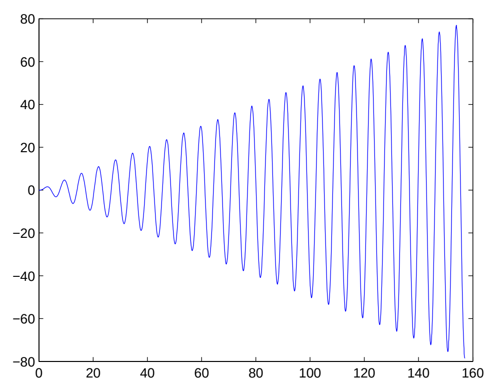


Figure 46: System in resonance

20 System of ODEs: the Predator-prey Model

In this section, we study a predator-prey model using a *system of first order ODEs*. The organization of the section is as follows.

- The model
- Solving the equations with `ode45`
- Plotting the solution

The model

Predator-prey is one of the fundamental problem of mathematical ecology. $F(t)$ and $R(t)$ denote the population of two species, one of which preys (F) on the other (R). For example they may be the number of foxes and rabbits. Without the prey, the predators will decrease and without the predator the prey will increase. A mathematical model was proposed in 1925 by Lotka and Volterra. It consists in a system of differential equations:

$$\frac{dR}{dt} = a_1 R - b_1\, F\, R$$
$$\frac{dF}{dt} = -a_2 F + b_2\, F\, R$$

Those are the predator prey equations. a_1 is the birth rate of the population R; a_2 is the death rate of population F. The FR terms in the two equations model the interaction of the two populations. The number of encounters between predator and prey is assumed to be proportional to the product of the populations. Since any such encounter tends to be good for the predator and bad for the prey, the sign of the FR term is negative in the first equation and positive in the second.
If we define the following vectors

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix};$$

and its derivative

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} R'(t) \\ F'(t) \end{pmatrix},$$

we can write the predator-prey model as

$$\frac{d\vec{y}}{dt} = f(t, \vec{y}),$$

where the function f is defined as follows

$$f(t, \vec{y}) = f\left(t, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} a_1 y_1 - b_1 y_1 y_2 \\ -a_2 y_2 + b_2 y_1 y_2 \end{pmatrix}.$$

Solving this equation with `ode45`

We can now solve this equation with `ode45`. The initial conditions are R=2 and F=1. The parameters are $a_1 = 1$, $b_1 = 0.5$, $a_2 = 0.75$, $b_2 = 0.25$. Here is the function used to model this ODE:

```
function yp = predator_prey(t,y)
% y(1) is the prey R
% y(2) is the predator F
yp = [y(1) - 0.5*y(1)*y(2); -0.75*y(2) + 0.25*y(1)*y(2)];
```

Now we call `ode45`:

```
[t,y] = ode45(@predator_prey,[0 30],[2 1]);
```

Plotting the solution

The first column of y contains R and the second contains F. We can plot the solution using:

```
plot(t,y(:,1),t,y(:,2),'Linewidth',2)
legend('Rabbit','Fox')
```

We can see the results in Figure 47

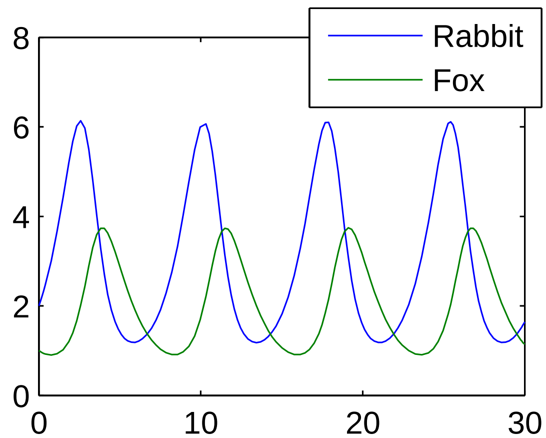


Figure 47: Population of predators and preys

Observe how the two curves are periodic. Even though this is a system of nonlinear equations, the behavior turns out to be suprisingly simple.
The code used is contained in the file "`predator.m`".
File "`predator.m`"

```
01 function predator
02     [t,y] = ode45(@predator_prey,[0 30],[2 1]);
03     plot(t,y(:,1),t,y(:,2))
04     legend('Rabbit','Fox')
05 end

06 function yp = predator_prey(t,y)
07 % y(1) is the prey R
08 % y(2) is the predator F
09     yp = [y(1) - 0.5*y(1)*y(2); -0.75*y(2) + 0.25*y(1)*y(2)];
10 end
```

We now comment on that code. In line 02, the parameters passed to `ode45` are the following:

- `@predator_prey` is a handle on the system of ODEs to consider.
- `[0 30]` in the interval of time considered.
- `[2 1]` are the number of preys (2) and predators (1) at time $t = 0$.

The function `ode45` returns the results as t and y. t is a vector with the values of the times at which the solution was approximated, and y is a *matrix* where the *first column* corresponds to the values of $y_1 = R$, and the *second column* corresponds to the values of $y_2 = F$.
With Matlab, in a matrix A, we select the i^{th} column with the command `A(:,i)`. Thus, to select the population of rabbits (or y_1) from the calculated solution y, we use the command `y(:,1)`. Similarly for the population of foxes (or y_2).

21 Solving Second Order Equations and Systems using Matlab

Assume that we want to solve a second order initial value problem of the form

$$y'' = f(x, y, y'), \quad \text{subject to} \tag{129}$$

$$y(0) = y_0, \quad \text{and} \tag{130}$$

$$y'(0) = y_0'. \tag{131}$$

21.1 Solving a Second Order Initial Value Problem

In order to solve this second order initial value problem, we formulate it as a system of first order ODEs. We introduce two functions y_1 and y_2 defined as follows

$$y_1' = y_2 \tag{132}$$

$$y_2' = f(x, y_1, y_2), \tag{133}$$

and we solve simultaneously both equations subject to

$$\begin{aligned} y_1(0) &= y_0 \\ y_2(0) &= y_0'. \end{aligned}$$

This is now a **system of first order ODEs**, which Matlab can solve. We can see that y_1 is the solution to our original initial value problem whose ODE is Equation 129. Observe that:

$$\begin{aligned} y'' &= y_2' \\ &= f(x, y_1, y_2) \\ &= f(x, y, y'). \end{aligned}$$

Thus $y = y_1$ satisfies Equation 129. The initial conditions are also clearly satisfied.

Example 42
We want to solve the initial value problem

$$\begin{aligned} y'' + xy' + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 2. \end{aligned}$$

The function f is therefore defined as

$$f(x, y, y') = -xy' - y.$$

Thus we get the system of ODEs to solve with Matlab

$$\begin{cases} y_1' = y_2 \\ y_2' = -xy_2 - y_1 \\ y_1(0) = y_0 \\ y_2(0) = y_0' \end{cases}$$

We use the same code used to solve the predator-prey system. If we plot the solution, we get Figure 48

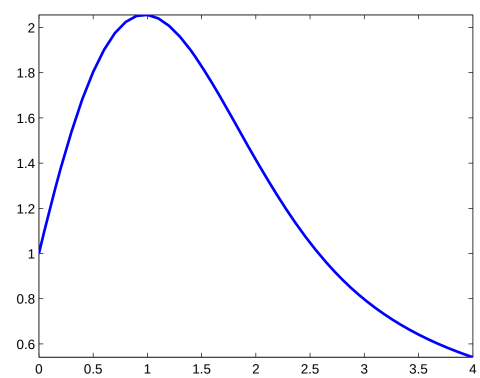


Figure 48: Solution to the system from Example 42

21.2 Solving Systems of Second Order ODEs

We now know how to solve a second order ODE using Matlab. What about systems of second order ODEs? For example in mechanics, the models used usually involve systems of second order ODEs (Newton's second law is an example of such systems). In those systems, t is the independent variable, and x, y and/or z are the functions to calculate.

Example 43

Consider the following system of second order ODEs

$$\begin{cases} x'' = -9x + 4y + x' + e^t - 6t^2 \\ y'' = 2x - 2y + 3t^2 \\ x(0) = x_0 \\ x'(0) = x_0' \\ y(0) = y_0 \\ y'(0) = y_0' \end{cases}$$

This type of systems typically represent a body moving in a plane. We can solve this with Matlab by introducing the following four functions

$$y_1 = x, \;\; y_2 = y, \;\; y_3 = x', \;\; y_4 = y'.$$

We can now define the system

$$\begin{cases} y_1' = y_3 \\ y_2' = y_4 \\ y_3' = -9y_1 + 4y_2 + y_3 + e^t - 6t^2 \\ y_4' = 2y_1 - 2y_2 + 3t^2 \\ y_1(0) = x_0 \\ y_2(0) = y_0 \\ y_3(0) = x_0' \\ y_4(0) = y_0' \end{cases}$$

We use the code from file "`ode2.m`" to plot the solution.

```
function ode2
    [t,y] = ode45(@f,[0 8],[0 -1 1 -20]);
    % t0 = 0; t_final = 8;
    % x(0) = 0; x'(0) = 1;
    % y(0) = -1; y'(0) = -20;
    % y is a matrix with 4 columns containing: x, y, x', y'

    plot(t,y(:,1),t,y(:,2),'Linewidth',2); axis tight;
    legend('x(t)','y(t)')
    % The first column y(:,1) contains x(t)
    % The second column y(:,2) contains y(t)
    figure
    plot(y(:,1),y(:,2),'Linewidth',2); axis tight;
    % This plot produces the trajectory of the body
end
```

```
function yp = f(t,y)
    yp = zeros(4,1);
    yp(1) = y(3);
    yp(2) = y(4);
    yp(3) = -9*y(1)+4*y(2)+y(3)+exp(t)-6*t.^2;
    yp(4) = 2*y(1)-2*y(2)+3*t.^2;
end
```

The results are shown in Figure 49 and 50.

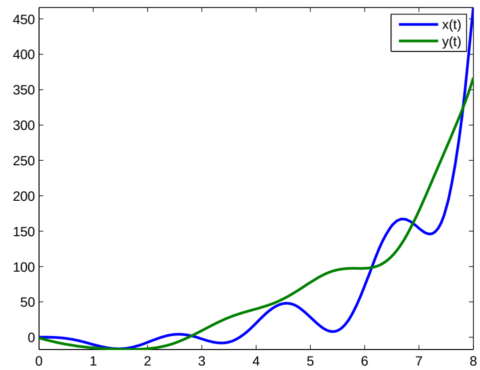


Figure 49: Solution to the system from Example 43

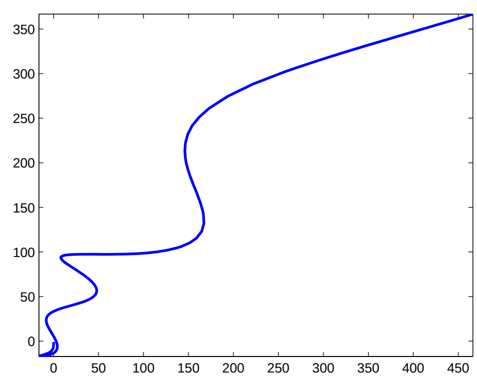


Figure 50: Trajectory of the body from Example 43

Let us now move to a more realistic example.

Example 44 Orbital Motion

In this example we consider a body orbiting a planet (or perhaps a planet orbiting the sun). In other terms, we consider one large mass placed at the origin of the euclidean space. We consider the motion of a second object, with a considerably smaller mass, subject only to the gravitational pull of the large mass.

We can prove that the trajectory of the body is contained in a plane. Further, we can use *polar coordinates* to describe the motion of the body. We will not derive the equations, but if we set the angle $\theta(t)$ and radius $r(t)$ as in Figure 51, we get the following system of equations.

$$\left.\begin{array}{l} r'' - (\theta')^2\, r = -\frac{4\pi^2}{r^2} \\ r\theta'' + 2r'\theta' = 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} r'' = (\theta')^2\, r - \frac{4\pi^2}{r^2} \\ \theta'' = -\frac{2r'}{r}\theta' \end{array}\right.$$

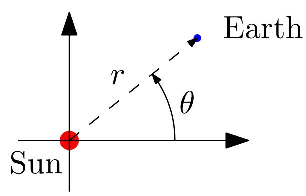


Figure 51: Polar coordinates

We can solve this with Matlab by introducing the following four functions

$$y_1 = r(t), \quad y_2 = \theta(t), \quad y_3 = r'(t), \quad y_4 = \theta'(t).$$

We can now define the system

$$\left\{\begin{array}{l} y_1' = y_3 \\ y_2' = y_4 \\ y_3' = y_1 y_4^2 - \frac{4\pi^2}{y_1^2} \\ y_4' = -2\frac{y_3}{y_1} y_4 \end{array}\right.$$

that we can solve with Matlab. You can find the code in the file "`orbit.m`".

```
function orbit(thetap0)
% We want to compute r(t) and theta(t).
%
% Because we have 2nd order ODEs, we need to introduce new variables and
% write a set of four differential equations with those new variables.
    option = odeset('RelTol',1e-6,'AbsTol',1e-6);
    [t,y] = ode45(@orbit_ode,[0 50],[2 0 0 thetap0],option);
% y contains: r, theta, r', theta'
% Note how the accuracy is changed using odeset and the variable option.

% The command polar can be used to plot a curve in polar coordinates
% (theta(t),r(t)).
    polar(y(:,2),y(:,1))
    title(sprintf('theta''(0) = %g',thetap0))
% This is a trick to print the value of theta'(0) in the title
end
```

```
function yp = orbit_ode(t,y)
% y = [r; theta; r'; theta']
    yp = zeros(4,1);
    yp(1) = y(3);
    yp(2) = y(4);
    yp(3) = y(1)*y(4)^2-4*pi^2/y(1)^2;
    yp(4) = -2*y(3)*y(4)/y(1);
end
```

In the code from file "`orbit.m`", the parameter `thetap0` is the value of the derivative of θ at $t = 0$. Depending on the value $\theta'(0)$ we observe a different behavior for the Earth's trajectory.

- Circular orbit: $\theta'(0) = \pi/\sqrt{2}$
  The trajectory of the Earth is then circular. We can see it in Figure 52.

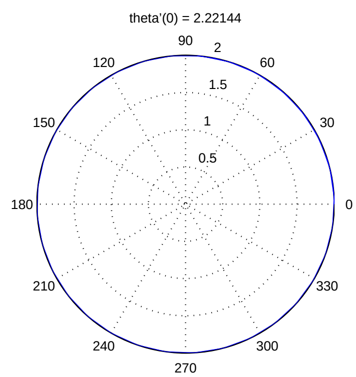


Figure 52: Circular orbit

- Elliptical orbit: $\theta'(0) < \pi/\sqrt{2}$
  The trajectory of the Earth is then elliptic. The Earth is closest to the Sun when $\theta = \pi$. We can see it in Figure 53.

- Elliptical orbit: $\pi > \theta'(0) > \pi/\sqrt{2}$
  The trajectory of the Earth is still elliptic. The Earth is farthest from the Sun when $\theta = \pi$. We can see it in Figure 54.

- Hyperbolic orbit: $\theta'(0) > \pi$
  The trajectory of the Earth is then Hyperbolic. The Earth will get farther and farther from the Sun and leave the solar system. This is show on Figure 55.

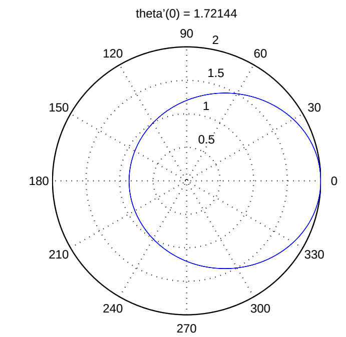


Figure 53: Elliptical orbit

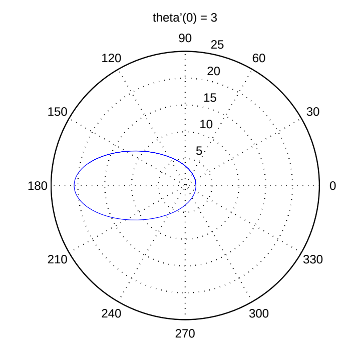


Figure 54: Elliptical orbit

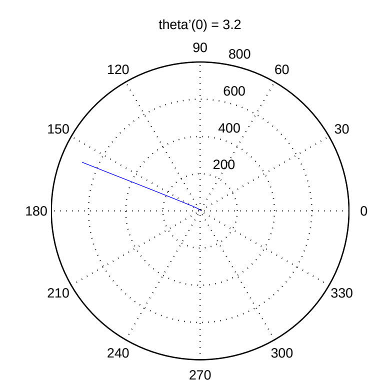


Figure 55: Hyperbolic orbit

21.3 Nested Functions

A very useful feature Matlab has is the hability to define *nested functions*. Imagine that, in the orbiting body example, we want to change $4\pi^2$ by a parameter G. We want to write a script that, when given the parameter G, would perform all the calculations and plot the resulting trajectory.
The problem is that `ode45` only accepts `t` and `y` as parameters. How can we pass an extra parameter G to `ode45`? This is an important and common issue in practice. We solve this problem using nested functions. We use the code from file "`orbit_nested.m`":

```
function orbit_nested(G)
% G is the gravitational constant
    option = odeset('RelTol',1e-6,'AbsTol',1e-6);
    [t,y] = ode45(@orbit_ode,[0 4],[2 0 0 pi/sqrt(2)],option);
% y contains: r, theta, r', theta'

    polar(y(:,2),y(:,1))
    title(sprintf('G = %g',G))

    function yp = orbit_ode(t,y) % This is a nested function!
        yp = zeros(4,1);
        yp(1) = y(3);
        yp(2) = y(4);
        yp(3) = y(1)*y(4)^2 - G/y(1)^2; % Variable G appears here
        % The nested function orbit_ode() can access G defined in
        % orbit_nested()
        yp(4) = -2*y(3)*y(4)/y(1);
    end
end
```

The idea behind nested functions is to define a main function (here `orbit_nested(G)`) where G is defined (here it is an input parameter). We then define the ODE function (`orbit_ode(t,y)`) inside the main function. Because this function is nested (e.g. is defined inside `orbit_nested`) it can access and use the value of G. If we change the value of G, we can see that G controls the magnitude of the gravitational force between the Sun and the Earth.

22 Power Series

The technique of power series is a powerful technique which can solve a wide range of problems modeled by ODEs. Since most functions have power series expansions, it is very general. We will focus, for simplicity, on linear second order homogeneous ODEs.

22.1 Power Series and Evaluating Usual Functions

Power series are very useful for calculating functions. All usual functions have known power series (given by their Taylor series expansion). Thus, if we truncate a power series, we can get a very good approximation to the real value of a function. That approximation is cheap to calculate as evaluating a truncated power series is equivalent to evaluating a polynomial.

22.2 Properties and Examples of Power Series

Recall that a power series of a function is defined around a given point x_0. An example of a power series of a function f infinitely differentiable at x_0 is its Taylor series expansion

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + ... \tag{134}$$

$$= \sum_{n=0}^{+\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \tag{135}$$

where $f^{(n)}(x_0)$ is the n^{th} derivative of f at x_0, and

$$n! = 1 \times 2 \times 3 \times 4 \times ... \times (n-1) \times n = \prod_{j=1}^{n} j.$$

Example 45 Power Series of Usual Functions
From Equation 135, we get the following power series for usual functions (around 0)

$$e^x = \sum_{n=0}^{+\infty} \frac{1}{n!} x^n$$

$$\cos(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sin(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

The next properties are extensively used in solving ODEs using power series.

Proposition 3 (Uniqueness of Power Series)**.** *Assume that we have two power series such that*

$$\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} b_n x^n,$$

then we have that, for all $n \geq 0$, $a_n = b_n$*.*

Proposition 4 (Power Series and Derivatives)**.** *Let f be a function such that its power series around 0 is given by*

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n, \tag{136}$$

then the power series of its derivative $f'(x)$ is given by

$$f'(x) = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n. \tag{137}$$

Proof: [We only prove the second result]
We take the derivative of Equation 136.

$$\begin{aligned}
f'(x) &= \sum_{n=0}^{+\infty} a_n \frac{dx^n}{dx}, \quad \text{since the term for } n=0 \text{ is zero, we have} \\
&= \sum_{n=1}^{+\infty} a_n \, n \, x^{n-1}, \quad \text{we set } m = n-1, \text{ then} \\
&= \sum_{m=0}^{+\infty} (m+1) a_{m+1} x^m, \quad \text{we now change back to } n \text{ as the dummy variable} \\
&= \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n
\end{aligned}$$

which proves the result. ■
An immediate consequence from the previous proposition is

Corollary 1 (Power Series and Second Derivative)**.** *Let f be a function where its power series is given by Equation 136, then the power series of its second derivative $f''(x)$ is given by*

$$f''(x) = \sum_{n=0}^{+\infty} (n+2)(n+1) a_{n+2} x^n. \tag{138}$$

Can you prove this result?

22.3 Power Series and ODEs

We start with an example.
Example 46
Consider the ODE $y'' + y = 0$. We know its general solution is given by

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

We look for a solution of the form

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

Thus, y is a solution implies

$$
\begin{aligned}
y'' + y &= \sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{+\infty} a_n x^n \\
&= \sum_{n=0}^{+\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n \\
&= 0.
\end{aligned}
$$

Thus we deduce that, for all $n \geq 0$, we have that

$$(n+2)(n+1)a_{n+2} + a_n = 0.$$

Now, for $n = 0$, we get $2a_2 + a_0 = 0$, thus, if we are given a_0, we can calculate

$$a_2 = -\frac{1}{2}a_0.$$

Now, for $n = 2$, we get that $(4)(3)a_4 + a_2 = 0$, thus, if we are given a_0, we can calculate

$$a_4 = -\frac{1}{12}a_2 = \frac{1}{24}a_0.$$

It is now clear that we can generalize this to all n even.
Similarly, we have that, for $n = 1$, $(3)(2)a_3 + a_1 = 0$. Thus, if we are given a_1, then we can calculate

$$a_3 = -\frac{1}{6}a_1.$$

Then in turn we can calculate

$$a_5 = -\frac{1}{20}a_3 = \frac{1}{120}a_1.$$

More generally, for all n odd, a_n only depends on a_1.
Hence we conclude that the general solution to the ODE $y'' + y = 0$ is given by

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - ...\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - ...\right),$$

We can recognize the power series expansion of sin and cos around 0! The solution can in fact be written in the form:

$$y(x) = a_0\cos(x) + a_1\sin(x).$$

In more complicated cases however we may obtain a power series which does not correspond to any known analytical function. In that case we have to leave the solution in the form of a power series expansion.

The coefficients a_0 and a_1 have a specific interpretation in terms of the initial conditions. From the definition of the power series of y, we get that $y(0) = a_0$. Similarly, from the expression of the power series of y', we get that $y'(0) = a_1$. Thus we can see that, in any initial value problem, we have that

$$
\begin{aligned}
a_0 &= y(0) = y_0 \\
a_1 &= y'(0) = y_0'.
\end{aligned}
$$

22.4 Airy's Equation

So far we have used the power series method for equations which we could solve by other means. Let's now consider a more difficult equation which can be solved only by the method of power series. Note that in this case, the solution will *not* be given by an known function (e.g. transcendental functions) but only by its power series expansion. This is the case of Airy's equation:

$$y'' - xy = 0 \tag{139}$$

There is no closed form for the general solution to such problem. We can nevertheless, heuristically, explore the behavior of the solution to Equation 139 using insights from known solutions to other ODEs.
Heuristically, if $x < 0$, the equation looks like $y'' + \omega^2 y = 0$. Thus we expect the solution to have sine characteristics: be oscillating and bounded. On the other hand, if $x > 0$, the equation looks like $y'' = \lambda^2 y$. Thus we expect the solution to have exponential characteristics: it should grow unbounded as x increases. We can see a plot of a Airy function in Figure 56

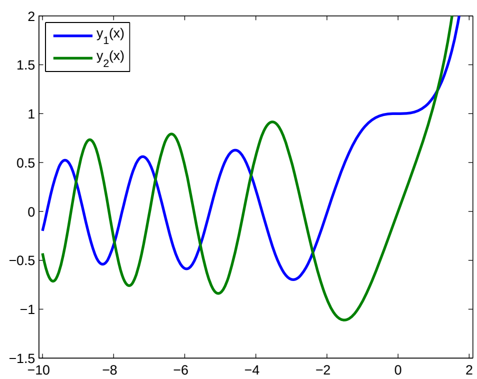


Figure 56: Airy function for different initial conditions

Let us look for a solution to the Airy equation using power series. Assume that the solution y is of the form $y(x) = \sum_{n=0}^{+\infty} a_n x^n$. Thus we deduce that

$$\begin{aligned}
xy(x) &= x\sum_{n=0}^{+\infty} a_n x^n \\
&= \sum_{n=0}^{+\infty} a_n x^{n+1}, \quad \text{set } m = n+1 \text{ and we get} \\
&= \sum_{m=1}^{+\infty} a_{m-1} x^m, \quad \text{change the dummy variable back to } n \text{ and get} \\
&= \sum_{n=1}^{+\infty} a_{n-1} x^n.
\end{aligned}$$

We know that its second derivative is given by

$$y''(x) = \sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2}x^n,$$

Since y is a solution to Equation 139, we must satisfy:

$$\begin{aligned} y'' - xy &= \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{+\infty} a_{n-1}x^n \\ &= (2)(1)a_2 + \sum_{n=1}^{+\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{+\infty} a_{n-1}x^n \\ &= 2\,a_2 + \sum_{n=1}^{+\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n \\ &= 0. \end{aligned}$$

Thus we have that, for all $n \geq 1$

$$\begin{aligned} &(n+2)(n+1)a_{n+2} - a_{n-1} = 0, \quad \text{we set } m = n-1 \text{ and get} \\ &(m+3)(m+2)a_{m+3} - a_m = 0. \end{aligned}$$

We now change back the dummy variable to n, and get that for all $n \geq 0$,

$$(n+3)(n+2)a_{n+3} = a_n. \tag{140}$$

Let's see what this equation tells us. If we set $n = 0$, we get $(3)(2)a_3 = a_0$:

$$a_3 = \frac{1}{6}a_0.$$

From $n = 3$, we get that $(6)(5)a_6 = a_3 = a_0/6$:

$$a_6 = \frac{1}{180}a_0.$$

More generally, we can see that the value of a_0 completely determines the value of all a_n's where n is any of the following numbers: $\{0, 3, 6, 9, 12, 15, 18, ...\}$, i.e. a multiple of 3.
If we now set $n = 1$, we get $(4)(3)a_4 = a_1$:

$$a_4 = \frac{1}{12}a_1.$$

From $n = 4$, we get that $(7)(6)a_7 = a_4 = a_1/12$:

$$a_7 = \frac{1}{504}a_1.$$

More generally, we can see that the value of a_1 completely determines the value of all a_n's where n is any of the following numbers: $\{1, 4, 7, 10, 13, 16, 19, ...\}$. These numbers are of the form $3k+1$.
If we now set $n = 2$, we know that $a_2 = 0$. But if we use Equation 140, we get $(5)(4)a_5 = a_2 = 0$. For $n = 5$, we get that $(8)(7)a_7 = a_5 = 0$. More generally, we can see that if n is any of the following numbers: $\{2, 5, 8, 11, 14, 17, 20, ...\}$, i.e. a number of the form $3k + 2$, then $a_n = 0$.
Gathering these results together, we obtain the following expression:

$$y(x) = a_0 \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + ...\right) + a_1 \left(x + \frac{1}{12}x^4 + \frac{1}{420}x^7 + ...\right).$$

Thus we see that the solution can be expressed as

$$y(x) = a_0 y_0(x) + a_1 y_1(x),$$

where y_0 and y_1 are defined by

$$\begin{aligned} y_0(x) &= 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + ... \\ y_1(x) &= x + \frac{1}{12}x^4 + \frac{1}{420}x^7 + ... \end{aligned}$$

These are not any known functions. They are simply given by their power series expansion.

23 Laplace Transform

This is the last method studied in this course to solve ODEs. We will focus on linear ODEs with constant coefficients. The general problem we consider is

$$ay'' + by' + cy = f(t). \tag{141}$$

The Laplace transform will prove very useful when the right hand side $f(t)$ is not continuous. This is often the case in mechanics and electric circuit design and analysis.

23.1 Definition

The idea behind the Laplace Transform is to define an operation that transforms a function into another function, and which simplifies the solution of ODEs.

Definition 2 Laplace Transform. Let f be a piecewise continuous function. Then we define its Laplace Transform $\mathcal{L}(f(t))(s) = F(s)$ as

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt \tag{142}$$

for all real numbers s for which this expression is well-defined.
The Laplace transform of a function $f(t)$ is itself a function $\mathcal{L}(f(t))(s) = F(s)$ of the variable s.
You will find many useful formulas on the book on pages 264–267.

Example 47 Let $f(t) = 1$. Then its Laplace transform is given by

$$F(s) = \int_0^{+\infty} e^{-st} dt.$$

Now we can see that, if $s \leq 0$, then the previous expression diverges. If $s > 0$, we have that

$$\begin{aligned} F(s) &= \left[-\frac{e^{-st}}{s} \right]_{t=0}^{+\infty} \\ &= -0 - \left(-\frac{1}{s} \right) \\ &= \frac{1}{s}. \end{aligned}$$

Thus we see that, for all $s > 0$, the Laplace transform of $f(t) = 1$ exists, and is equal to $1/s$.

Example 48
Let β be a given number, and define $f(t) = \exp(\beta t)$. Then its Laplace transform is given by

$$\begin{aligned} F(s) &= \int_0^{+\infty} e^{\beta t} e^{-st} dt \\ &= \int_0^{+\infty} e^{-(s-\beta)t} dt \end{aligned}$$

Now we can see that, if $s - \beta \leq 0$, then the previous expression diverges. If $s > \beta$:

$$\begin{aligned} F(s) &= \left[-\frac{e^{-(s-\beta)t}}{s-\beta} \right]_{t=0}^{+\infty} \\ &= \frac{1}{s-\beta}. \end{aligned}$$

For all $s > \beta$, the Laplace transform of $f(t) = \exp(\beta t)$ exists, and is equal to $1/(s - \beta)$.

23.2 Properties of the Laplace Transform

Proposition 5 (Linearity). *The Laplace transform is a linear operation. In other words, let f and g be two given functions, and let α and β be two real numbers, then we have that*

$$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha \mathcal{L}(f(t)) + \beta \mathcal{L}(g(t)).$$

See 3rd formula page 264. This property is naturally inherited from the linearity of the integration operation.

Assume you are given a function $F(s)$ such that, for some function $f(t)$, $F(s)$ is the Laplace transform of $f(t)$,

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt.$$

Then we say that $f(t)$ is the *inverse transform* of $F(s)$. The inverse Laplace transform is denoted by $\mathcal{L}^{-1}(F(s))$. We can prove that:

Proposition 6 (Inverse Transformation)**.** *The inverse Laplace transform $\mathcal{L}^{-1}$ is a linear operation.*

Instead of providing an analytic procedure to invert Laplace transforms, we use tables (such as that of the textbook, pages 264–267) of usual/known Laplace transforms.
The following is an important result for calculating inverse Laplace transforms. We will need it later to solve ODEs using the Laplace transform.

Proposition 7 (s-shift)**.** *For any function $f(t)$ and any real number β, we have that*

$$\mathcal{L}\left(f(t)e^{\beta t}\right)(s) = \mathcal{L}(f)(s-\beta).$$

See page 264, 4th formula.
Proof: We directly calculate $\mathcal{L}\left(f(t)e^{\beta t}\right)(s)$:

$$\begin{aligned}\mathcal{L}\left(f(t)e^{\beta t}\right)(s) &= \int_0^{+\infty} f(t)e^{\beta t}e^{-st}dt \\ &= \int_0^{+\infty} f(t)e^{-(s-\beta)t}dt \\ &= \mathcal{L}(f)(s-\beta).\end{aligned}$$

■

We now give the most important property of the Laplace transform with respect to solving linear ODEs.

Proposition 8 (Transformation of Derivatives)**.** *Let f be a differentiable function. Then the Laplace transform of its derivative $f'(t)$ is given by*

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0). \tag{143}$$

See page 264, 6th formula.
Proof:
If we write the definition of $\mathcal{L}(f'(t))$, we get

$$\mathcal{L}(f'(t)) = \int_0^{+\infty} f'(t)e^{-st}dt.$$

If we set $u'(t) = f'(t)$ and $v(t) = \mathrm{e}^{-st}$, from the integration by parts formula, we get

$$\begin{aligned}\mathcal{L}(f'(t)) &= \left[f(t)e^{-st}\right]_{t=0}^{+\infty} - \int_0^{+\infty} f(t)\,(-s)\,e^{-st}dt \\ &= \left[f(t)e^{-st}\right]_{t=0}^{+\infty} + s\int_0^{+\infty} f(t)e^{-st}dt.\end{aligned}$$

If we assume that, as t goes to $+\infty$, $f(t)e^{-st} \to 0$, then:

$$\mathcal{L}(f'(t)) = 0 - f(0) + s\int_0^{+\infty} f(t)e^{-st}dt = s\mathcal{L}(f(t)) - f(0).$$

■

A immediate consequence of Proposition 8 is that

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0). \tag{144}$$

See page 264, 7th formula.

23.3 Laplace Transform of Linear ODEs

The key observation is that from Propositions 5 and 8, we see that the Laplace transform, when applied to a linear ODE with constant coefficients, transforms the ODE into an **algebraic equation.** This algebraic equation can then be solved very easily. Applying the inverse Laplace transform yields the solution to the ODE. The inverse transform is usually the most complex operation in this method.

Let us illustrate this process with an example.

Example 49

Assume we want to solve the following initial value problem

$$\begin{aligned} y'' + 3y' + 2y &= 0, \quad \text{subject to} \\ y(0) &= 0, \quad \text{and} \\ y'(0) &= 2. \end{aligned}$$

We take the Laplace transform of the ODE and get

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = 0.$$

where $Y = \mathcal{L}(y)$. This is an algebraic equation in Y. Using the initial conditions:

$$s^2Y(s) - 2 + 3sY(s) + 2Y(s) = 0.$$

Solving for Y:

$$Y(s) = \frac{2}{s^2 + 3s + 2}.$$

We now need to find the inverse Laplace transform of Y. This is usually the most difficult step in the method. Look at the inverse transforms on page 265 and 266. Partial fractions are typically needed:

$$\frac{2}{s^2+3s+2} = \frac{2}{s+1} - \frac{2}{s+2},$$

Given that the inverse Laplace transform is linear, we have that

$$\mathcal{L}^{-1}(Y) = 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right).$$

We use the result from Example 48:

$$\mathcal{L}^{-1}\left(\frac{1}{s-\beta}\right) = e^{\beta t}.$$

Thus we conclude that the solution to the initial value problem is

$$y(t) = 2e^{-t} - 2e^{-2t}.$$

The previous example shows that, when using the Laplace transform, the initial conditions are included in the algebraic equation that defines the Laplace transform. This is always the case and is one of the advantages of the method.
Let us illustrate the general procedure by considering Equation 141 and the initial conditions $y(0) = 0$ and $y'(0) = 0$. Then, if we set $\mathcal{L}(f(t)) = F(s)$, we have that the Laplace transform of the solution to the initial value problem is given by

$\mathcal{L}\left(ay'' + by' + cy\right) = \mathcal{L}\left(f(t)\right),$ we use Proposition 5 to get:

$a\mathcal{L}\left(y''\right) + b\mathcal{L}\left(y'\right) + c\mathcal{L}\left(y\right) = F(s),$ we use Proposition 8 to get:

$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = F(s).$

From the initial conditions, we get:

$$as^2Y(s) + bsY(s) + cY(s) = F(s).$$

We can rewrite this as

$$Y(s) = \frac{F(s)}{as^2+bs+c} \tag{145}$$

Example 50 Assume we want to solve the following initial value problem

$$\begin{aligned} y'' + y &= \sin(2t), \quad \text{subject to} \\ y(0) &= 2, \quad \text{and} \\ y'(0) &= 1. \end{aligned}$$

We take the Laplace transform of the ODE and get

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}.$$

Applying the initial conditions:

$$s^2Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2+4},$$

which can be rearranged as

$$\begin{aligned} Y(s) &= \frac{1}{s^2+1}\left(\frac{2}{s^2+4} + 2s + 1\right) \\ &= \frac{2 + (2s+1)(s^2+4)}{(s^2+1)(s^2+4)} \\ &= \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} \end{aligned}$$

Now we look for a partial fraction decomposition of the form

$$\begin{aligned} Y(s) &= \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} \\ &= \frac{(as+b)(s^2+4) + (cs+d)(s^2+1)}{(s^2+1)(s^2+4)} \\ &= \frac{as^3 + bs^2 + 4as + 4b + cs^3 + ds^2 + cs + d}{(s^2+1)(s^2+4)} \\ &= \frac{(a+c)s^3 + (b+d)s^2 + (4a+c)s + 4b + d}{(s^2+1)(s^2+4)} \end{aligned}$$

Thus we must have:

$$(a+c)s^3 + (b+d)s^2 + (4a+c)s + 4b + d = 2s^3 + s^2 + 8s + 6$$

which is equivalent to the following system of equations

$$\begin{cases} a + c = 2 \\ b + d = 1 \\ 4a + c = 8 \\ 4b + d = 6 \end{cases}$$

whose solution is $(a, b, c, d) = (2, 5/3, 0, -2/3)$. Hence the Laplace transform of the solution is given by

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4},$$

which leads to

$$y(t) = \mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{5/3}{s^2+1}\right) - \mathcal{L}^{-1}\left(\frac{2/3}{s^2+4}\right).$$

From the tables in the textbook, we see that

$$\mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) = 2\cos(t)$$
$$\mathcal{L}^{-1}\left(\frac{5/3}{s^2+1}\right) = \frac{5}{3}\sin(t)$$
$$\mathcal{L}^{-1}\left(\frac{2/3}{s^2+4}\right) = \frac{1}{3}\sin(2t)$$

Finally the solution to the initial value problem is:

$$y(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t).$$

23.4 Discontinuous Function $f(t)$

Consider again the general problem:

$$\begin{aligned} ay'' + by' + cy = f(t), &\quad \text{subject to} \\ y(0) = y_0, &\quad \text{and} \\ y'(0) = y_0'. & \end{aligned}$$

For general initial conditions, we can derive the following equation for $Y(s) = \mathcal{L}(y)(s)$:

$$Y(s) = \frac{F(s)}{as^2+bs+c} + \frac{ay_0 s + ay_0' + by_0}{as^2+bs+c}. \tag{146}$$

In order to find the solution to the given initial value problem, it suffices to calculate

$$\mathcal{L}^{-1}\left(\frac{F(s)}{as^2+bs+c}\right), \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{ay_0 s + ay_0' + by_0}{as^2+bs+c}\right).$$

The second term can be calculated using simple partial fractions and the Laplace transform tables from the textbook.

For the first term, if we assume that the Laplace transform of $f(t)$ is a *rational function* (which is the case for most functions), then $F(s) = \mathcal{L}(f)$ is a polynomial in s. From the Laplace transform tables in the textbook and using partial fractions, the inverse transform of the first term can be computed.

What if the right hand side $f(t)$ in Equation 141 is a discontinuous function? For instance, think of a pickup being charged. The force applied to the rear suspension of the pickup may look, as a function of time, as Figure 57.

Or imagine what happens to the intensity of the current in an electrical circuit as we toggle the switch from off to on to off again. The current intensity is shown in Figure 58.

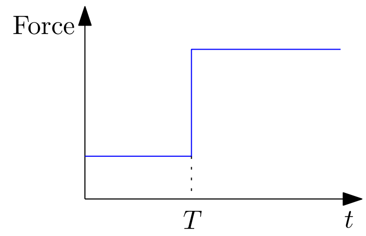


Figure 57: Truck loaded at time $t = T$

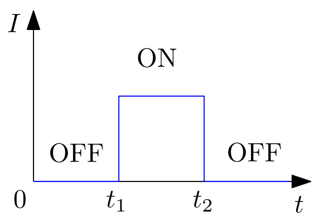


Figure 58: Current's intensity. Switch operating at t_1 and t_2

How do we solve an ODE when we have a discontinuous function? We start by considering the simplest discontinuous function, called the *Heaviside function*.

Definition 3 Heaviside function
The Heaviside function is defined as

$$U(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0 \end{cases}$$

For a real number $c > 0$, in Figure 59, we plot the function $U(t-c)$.

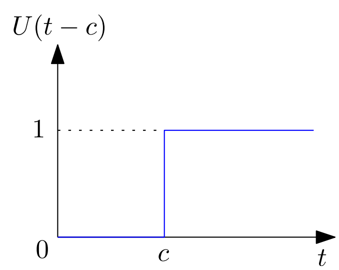


Figure 59: Heaviside function $U(t-c)$

With U we can model a discontinuity happening at any time. Let us now calculate the Laplace transform of $U(t-c)$:

$$\begin{aligned} \mathcal{L}\left(U(t-c)\right)(s) &= \int_0^{+\infty} U(t-c)e^{-st}dt \\ &= \int_0^c 0 \cdot e^{-st}dt + \int_c^{+\infty} 1 \cdot e^{-st}dt \\ &= \frac{e^{-cs}}{s} \end{aligned}$$

We can also prove the following more general result:

Proposition 9 (t-shift)**.** *The Laplace transform of* $U(t-c)f(t-c)$ *is given by*

$$\mathcal{L}\left(U(t-c)f(t-c)\right)(s) = e^{-cs}\,\mathcal{L}(f(t))(s) = e^{-cs}\,F(s).$$

See 12th formula page 264.

Example 51

We want to calculate the Laplace transform of the following function

$$V(t) = \begin{cases} \sin(t), & \text{if } t \in \left[0, \frac{\pi}{4}\right] \\ \sin(t) + \cos\left(t - \frac{\pi}{4}\right), & \text{if } t > \frac{\pi}{4} \end{cases}$$

We can see a plot of V in Figure 60.

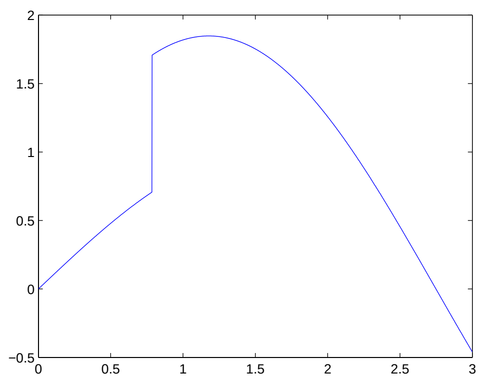


Figure 60: Discontinuous function $V(t)$

What is $\mathcal{L}(V(t))(s)$? Let's use the Heaviside function for this calculation. We first need to write V in terms of the Heaviside function:

$$V(t) = \sin(t) + U\left(t - \frac{\pi}{4}\right)\cos\left(t - \frac{\pi}{4}\right).$$

Thus, using the linearity of the Laplace transform, we have that

$$\mathcal{L}(V(t))(s) = \mathcal{L}(\sin(t))(s) + \mathcal{L}\left(U\left(t - \frac{\pi}{4}\right)\cos\left(t - \frac{\pi}{4}\right)\right)(s).$$

We get the first term from the textbook's tables. The second term, from the t-shift result, by setting $c = \pi/4$ and $f(t) = \cos(t)$, is:

$$\mathcal{L}\left(U\left(t - \frac{\pi}{4}\right)\cos\left(t - \frac{\pi}{4}\right)\right)(s) = e^{-\frac{\pi}{4}s}\,\mathcal{L}(\cos(t)).$$

From the textbook's tables:

$$\mathcal{L}(V(t))(s) = \frac{1}{s^2+1} + \frac{s\,e^{-\frac{\pi s}{4}}}{s^2+1}$$

Example 52
Consider $V(t) = U(t - \pi/4)\cos(t)$. How do we calculate the Laplace transform of V? In order to use the t-shift result, we need to put $V(t)$ in the form

$$V(t) = U\left(t - \frac{\pi}{4}\right) f\left(t - \frac{\pi}{4}\right),$$

thus, in this case we need to set

$$f\left(t - \frac{\pi}{4}\right) = \cos(t)$$

so that we get

$$f(t) = \cos\left(t + \frac{\pi}{4}\right).$$

Thus we have that

$$\mathcal{L}(V(t))(s) = e^{-\frac{\pi}{4}s}\,F(s) = e^{-\frac{\pi}{4}s}\,\mathcal{L}\left(\cos\left(t + \frac{\pi}{4}\right)\right).$$

Now we see that

$$f(t) = \cos(t)\cos\left(\frac{\pi}{4}\right) - \sin(t)\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(\cos(t) - \sin(t)\right),$$

and

$$F(s) = \frac{\sqrt{2}}{2}\mathcal{L}(\cos(t)) - \frac{\sqrt{2}}{2}\mathcal{L}(\sin(t)).$$

From the textbook tables we get

$$\mathcal{L}(V(t))(s) = e^{-\frac{\pi}{4}s}\,\frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1} - \frac{1}{s^2+1}\right)$$

Example 53 Hat function
We consider the following function

$$h(t) = \begin{cases} 0, & \text{if } t \leq \pi \\ 1, & \text{if } t \in (\pi, 2\pi) \\ 0, & \text{if } t \geq 2\pi \end{cases}$$

which is called the *hat function*. We can see its plot in Figure 61.
We can see that, if we can calculate the Laplace transform of the hat function, given that all discontinuous functions can be written as sums and products of hat and Heaviside functions, we can calculate the Laplace transform of any discontinuous function.
The hat function can be written using Heaviside functions as

$$h(t) = U(t - \pi) - U(t - 2\pi),$$

Thus its Laplace transform $H(s)$ is given by

$$H(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}.$$

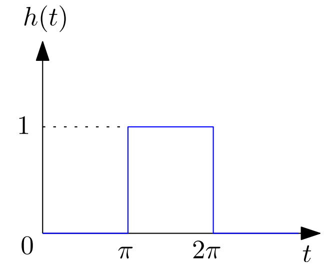


Figure 61: Hat function $h(t)$

23.5 Solving ODEs with a discontinuous forcing term

We present two examples of the use of Laplace transforms to solve initial value problems with a discontinuous forcing term.

Example 54
We are interested in solving the following initial value problem:

$$\begin{aligned} 2y'' + y' + 2y &= f(t), \quad \text{subject to} \\ y(0) = 0,\ y'(0) &= 0, \quad \text{and} \\ f(t) &= \begin{cases} 0 \text{ for } 0 \le t \le 5, \text{ and } t > 20 \\ 1 \text{ for } 5 < t < 20 \end{cases} \end{aligned}$$

Using the Laplace transform, the steps are as follows:

- We start by expressing $f(t)$ using the Heaviside function. Here we see that f is a hat function, hence
$$f(t) = U(t-5) - U(t-20).$$
We can thus easily calculate its Laplace transform
$$\mathcal{L}(U(t-5) - U(t-20)) = \mathcal{L}(U(t-5)) - \mathcal{L}(U(t-20)) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}.$$

- Next we calculate the Laplace transform of the ODE
$$\mathcal{L}(2y'' + y' + 2y) = \mathcal{L}(f(t)).$$
This yields
$$\begin{aligned} Y(s)(2s^2 + s + 2) &= \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}, \quad \text{or} \\ Y(s) &= \frac{e^{-5s}}{s(2s^2+s+2)} - \frac{e^{-20s}}{s(2s^2+s+2)} \\ &= e^{-5s}H(s) - e^{-20s}H(s) \end{aligned}$$

- We now calculate the inverse transform of $H(s)$. Given that $2s^2+s+2=0$ has no real solutions, partial fractions gives:

$$H(s) = \frac{1}{s(2s^2+s+2)} = \frac{1/2}{s} - \frac{1/2+s}{2s^2+s+2}$$

Now, completing the square yields $2s^2+s+2 = 2\left[(s+\frac{1}{4})^2+\frac{15}{16}\right]$. Hence we have

$$\frac{1/2+s}{2s^2+s+2} = \frac{(s+1/4)+1/4}{2\left[(s+\frac{1}{4})^2+\frac{15}{16}\right]} = \frac{1}{2}\frac{(s+1/4)}{(s+\frac{1}{4})^2+\frac{15}{16}} + \frac{1/8}{(s+\frac{1}{4})^2+\frac{15}{16}}.$$

The inverse transforms are:

$$\mathcal{L}^{-1}\left(\frac{(s+1/4)}{(s+\frac{1}{4})^2+\frac{15}{16}}\right) = e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right)$$
$$\mathcal{L}^{-1}\left(\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}\right) = \frac{4}{\sqrt{15}}\,e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

Thus we get

$$\mathcal{L}^{-1}\left(\frac{1/2+s}{2s^2+s+2}\right) = \frac{1}{2}\,e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right) + \frac{1}{2\sqrt{15}}\,e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

Which implies that

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right) \tag{147}$$

- Finally, we calculate the inverse transform of $Y(s)$. For this, we will need to use the t-shifting theorem. We have that

$$Y(s) = e^{-5s}H(s) - e^{-20s}H(s),$$

thus, using the t-shifting theorem, we get

$$\mathcal{L}^{-1}\left(e^{-5s}H(s)\right) = U(t-5)h(t-5)$$
$$\mathcal{L}^{-1}\left(e^{-20s}H(s)\right) = U(t-20)h(t-20).$$

The solution to the initial value problem is therefore given by:

$$y(t) = U(t-5)h(t-5) - U(t-20)h(t-20).$$

with $h(t)$ given by Equation 147.

Example 55

We are interested in solving the following initial value problem

$$
\begin{aligned}
y'' + 4y &= f(t), \quad \text{subject to} \\
y(0) = 0,\ y'(0) &= 0, \quad \text{and} \\
f(t) &= \begin{cases} 0 \text{ for } 0 \le t < 5 \\ \frac{t-5}{5} \text{ for } 5 \le t < 10 \\ 1 \text{ for } t \ge 10 \end{cases}
\end{aligned}
$$

Again, using the Laplace transform:

- We start by expressing $f(t)$ using the Heaviside function. Here we see that we will use the hat function to calculate f. We begin by getting an expression for $5 \le t < 10$. Let

$$f_1(t) = [U(t-5) - U(t-10)]\left[\frac{t-5}{5}\right].$$

For $t \ge 10$, let

$$f_2(t) = U(t-10).$$

Then we have that $f(t) = f_1(t) + f_2(t)$. Thus

$$
\begin{aligned}
f(t) &= [U(t-5) - U(t-10)]\left[\frac{t-5}{5}\right] + U(t-10) \\
&= U(t-5)\Big(\frac{t-5}{5}\Big) + U(t-10)\Big(1 - \frac{t-5}{5}\Big) \\
&= U(t-5)\Big(\frac{t-5}{5}\Big) - U(t-10)\Big(\frac{t-10}{5}\Big)
\end{aligned}
$$

We can thus easily calculate the Laplace transform of f

$$\mathcal{L}(f(t)) = \mathcal{L}\left(U(t-5)\Big(\frac{t-5}{5}\Big)\right) - \mathcal{L}\left(U(t-10)\Big(\frac{t-10}{5}\Big)\right)$$

which, from the t-shifting theorem yields

$$
\begin{aligned}
F(s) &= \frac{e^{-5s}}{5}\mathcal{L}(t) - \frac{e^{-10t}}{5}\mathcal{L}(t) \\
&= \frac{e^{-5s}}{5}\frac{1}{s^2} - \frac{e^{-10s}}{5}\frac{1}{s^2}
\end{aligned}
$$

- Next we calculate the Laplace transform of the ODE

$$\mathcal{L}(y'' + 4y) = \mathcal{L}(f(t)).$$

We get:

$$
\begin{aligned}
Y(s)(s^2+4) &= \frac{e^{-5s}}{5}\frac{1}{s^2} - \frac{e^{-10s}}{5}\frac{1}{s^2}, \quad \text{or} \\
Y(s) &= \frac{e^{-5s}}{5}\frac{1}{s^2(s^2+4)} - \frac{e^{-10s}}{5}\frac{1}{s^2(s^2+4)} \\
&= \frac{e^{-5s}}{5}H(s) - \frac{e^{-10s}}{5}H(s)
\end{aligned}
$$

- We now calculate the inverse transform of $H(s)$. Given that $s^2+4=0$ has no real solutions, partial fractions gives:

$$H(s) = \frac{1}{s^2(s^2+4)} = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

  Now we have

$$\begin{aligned} \mathcal{L}^{-1}\left(H(s)\right) &= \mathcal{L}^{-1}\left(\frac{1/4}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1/4}{s^2+4}\right) \\ h(t) &= \frac{1}{4}\,t - \frac{1}{8}\sin(2t) \end{aligned} \tag{148}$$

- Finally, we calculate the inverse transform of $Y(s)$. For this, we will need to use the t-shifting theorem. We have that

$$Y(s) = \frac{e^{-5s}}{5}H(s) - \frac{e^{-10s}}{5}H(s)$$

  thus, using the t-shifting theorem, we get

$$\begin{aligned} \mathcal{L}^{-1}\left(e^{-5s}H(s)\right) &= U(t-5)h(t-5) \\ \mathcal{L}^{-1}\left(e^{-10s}H(s)\right) &= U(t-10)h(t-10). \end{aligned}$$

The solution is therefore equal to:

$$y(t) = U(t-5)\,\frac{h(t-5)}{5} - U(t-10)\,\frac{h(t-10)}{5}$$

with h given by Equation 148.

This concludes our lecture notes on ordinary differential equations. We hope you liked the class!